CHAPTER-2
PRELIMINARIES AND BASIC CONCEPTS

2.1 Introduction:

Although most of the concepts, notations, definitions and results are standardized by now, for the sake of completeness, in this chapter, we collect those definitions and results and obtain some pre-requisites which will be used in the subsequent chapters. Here, the results are mentioned without proof and can be seen in the papers referred to. In this chapter we collect the basic definitions and obtain some pre-requisites, which will be used in the subsequent chapters.

2.2 Some Definitions, Notations and Results:

Definition 2.2.1 [103]: Let X be an arbitrary nonempty set. A fuzzy set in X is a mapping from X to the closed unit interval $I = [0,1]$, that is, an element of I^X. A fuzzy point (some times called a fuzzy singleton) is a fuzzy set in X if it takes value 0 for all y∈ X except one, say x∈ X. If its value at x is λ $(0 < \lambda \leq 1)$. We denote the fuzzy point P_x^λ, where the point x is called its support. We denote by S(X) the collection of all fuzzy points in X. the fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X respectively

Definition 2.2.2 [103]: A fuzzy point P_x^λ is said to be contained in a fuzzy set A or said to belong to A, denoted by $p_x^\lambda \in$ A if and only if A(x) ≥ λ. Every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A. That is, if A(x) is not zero for x∈ X, then A(x) = sup{λ: p_x^λ is fuzzy point and 0< λ ≤ 1}. For two fuzzy sets A and B in X, we write $A \subseteq B$ (or $B \supseteq A$) iff $A(x) \leq B(x)$ for all $x \in$ X and we then say that A is a subset of B; the negation of such a statement is denoted by $A \not\leq B$

Definition 2.2.3 [103]: Let X be a set A and B be two fuzzy subsets of X. A is said to be equal to B if and if A(x) = B(x) for every x$\in$ X. It is denoted by A= B

Definition 2.2.4 [103]: Let A, B $\in I^X$. We define the following fuzzy sets:

(i) $A \bigcap B \in I^X$, by $(A \bigcap B)(x) = \min \{ A(x), B(x) \}$ for each x$\in$ X

(ii) $A \bigcup B \in I^X$, by $(A \bigcup B)(x) = \max \{ A(x), B(x) \}$ for each x$\in$ X

(iii) $A^c \in I^X$, by $A^c(x) = 1 - A(x)$ for each x$\in$ X

Definition 2.2.5 [103]: Let $\{A_\alpha : \alpha \in \Lambda\}$, where Λ is an index set, be a family of fuzzy sets in X. The union $\bigcup_{\alpha\in\Lambda} A_\alpha$ and the intersection $\bigcap_{\alpha\in\Lambda} A_\alpha$ of the collection are the fuzzy sets given by $\bigcup_{\alpha\in\Lambda} A_\alpha = \sup \{ A_\alpha(x) : \alpha \in \Lambda \}$, x $\in$ X and $\bigcap_{\alpha\in\Lambda} A_\alpha = \inf \{ A_\alpha(x) : \alpha \in \Lambda \}$, x $\in$ X respectively.

Definition 2.2.6 [21]: Let T $\subseteq I^X$ satisfy the following three condition:

(i) X, $\varnothing \in I^X$

(ii) If A, B $\in$ T, then $A \bigcap B \in$ T

(iii) If $\{A_\alpha : \alpha \in \Lambda\} \subseteq$ T, then $\bigcup_{\alpha\in\Lambda} A_\alpha \in T$.

Then T is called a fuzzy topology on X and (X, T) is called a topological space (fts, in short). The elements of T are called open fuzzy sets in X and their complements are called fuzzy closed sets in X.

Definition 2.2.7 [6]: The closure and interior of a fuzzy set A in an fts (X, T) are denoted by clA and $\operatorname{int} A$ respectively and are defined as follows:

$$clA = \bigcap\{B : A \subseteq B \text{ and } B' \in T\}$$

$$\operatorname{int} A = \bigcup\{B : B \subseteq A \text{ and } B \in T\}$$

Result 2.2.8 [6]: A fuzzy point $p_x^\lambda \in$ cl(A) if and only if each Q-neighborhood of p_x^λ is quasi-concident with A.

Result 2.2.9 [6]: A fuzzy point $p_x^\lambda \in$ Int(A) if and only if p_x^λ has a neighborhood contained in A.

Result 2.2.10 [6]: For a fuzzy set A in an fts X we have:

(i) $1 - \operatorname{int} A = cl(1 - A)$

(ii) $1 - clA = \operatorname{int}(1 - A)$

(iii) A is fuzzy open (closed) iff $A = \operatorname{int}(resp.A = clA)$

Result 2.2.11 [6]: For a family $\{A_\alpha : \alpha \in \Lambda\}$ (Λ is an index set) fuzzy sets in X we have

(i) $\bigcup_{\alpha\in\Lambda} clA_\alpha \subseteq cl \bigcup_{\alpha\in\Lambda} A_\alpha$,equality holds if Λ is finite

(ii) $\bigcup_{\alpha\in\Lambda} \operatorname{int} A_\alpha \subseteq \operatorname{int} \bigcup_{\alpha\in\Lambda} A_\alpha$

Definition 2.2.12 [84]: Let (X, T) be a fts and A, B $\in I^X$. Then

(i) A is said to be quasi-coincident with B, denoted by A q B, if and only if there exists of x$\in$ X such that A(x) $\geq$ B^c(x) i.e. A(x) + B(x) > 1. If A is not quasicoincident with B, then we write A $\bar{q}$ B

(ii) A is called a Q-neighbourhood of $p_x^\lambda \in$ S(X) if and only if there exists $U \in T$ such that P_x^λ q U and U $\subseteq$ A. The class of all open Q-neighborhoods of P_x^λ is denoted by $N^Q(P_x^\lambda)$

Result 2.2.13 [84]: Let A and B be fuzzy subsets of (X,T). Then A $\subseteq$ B if and only if A and B^c are not quasi-concident; Particularly $p_x^\lambda \in$ A if and only p_x^λ is not quasi-coincident with A^c.

Result 2.2.14 [51]: Let A and B be fuzzy sets in an fts X. Then

(i) If $A \cap B = 0_X$, then $A \overline{q} B$

(ii) If $AqB \Rightarrow A \cap B \neq 0_X$

(iii) $A \subseteq B \Leftrightarrow P_x^{\lambda}$ q B for each P_x^{λ} q A

Result 2.2.15 [84]: Let (X,T) be a fts and A$\in I^X$. Then

(i) If $AqB \Rightarrow A \cap B \neq 0_X$

(ii) A is open iff $\forall$ P_x^{λ} q A, $\exists$ U$\in N^Q(P_x^{\lambda})$ such that U$\subseteq$A.

(iii) For each U$\in T$, U q A iff U q cl(A)

Definition 2.2.16 [84]: A fuzzy point p_x^{λ} in an fts X is called a fuzzy cluster point of a fuzzy set A in X every Q-nbd (or equivalently, every open q-nbd) of p_x^{λ} is q-coincident with A.

Result 2.2.17 [84]: For a fuzzy set A and a fuzzy point p_x^{λ} in an fts X, $p_x^{\lambda} \in clA$ iff p_x^{λ} is a fuzzy cluster point of A.

Result 2.2.18 [84]: For any two fuzzy sets A and B in an fts X of which B is fuzzy open, $A\overline{q}B \Rightarrow clA\overline{q}B$.

Definition 2.2.19 [40, 93]: The union of all fuzzy θ-open (resp. δ-open) sets contained in a fuzzy set A in an fts (X, T) is called the fuzzy θ-interior (resp. δ-interior) of A, to be denoted by θ-$\operatorname{int} A$ (resp. $\delta \operatorname{int} A$).

Definition 2.2.20 [56]: A fuzzy filter on X is a nonempty subset $\Phi \subseteq I^X$ such that

(i) $\varnothing \notin \Phi$

(ii) If A$\in\Phi$ and A $\subseteq$ B, then B$\in\Phi$.

(ii) If A, B $\in \Phi$, then A $\cap$ B$\in\Phi$.

The class of all fuzzy filter on X will be denoted by FL(I^X).

Definition 2.2.21 [52]: Let B be a nonempty family of subsets of I^X. Then B is called a base for a fuzzy filter on X (or a fuzzy filter-base) if the following two conditions are satisfied:

(i) $\varnothing \notin$ B

(ii) If A, B $\in$ B, then $A \cap B \in$ B

Definition 2.2.22 [56]: A fuzzy filter base on X is a nonempty subset $\Phi \subset I^X$ such that

(i) $\varnothing \notin \Phi$

(ii) If A, B $\in \Phi$, then $\exists$ C$\in \Phi$ such that $C \subseteq A \cap B$

The class of all fuzzy filter bases on X will be denoted by FLB(I^X).

The fuzzy filter Φ generated by B is defined by $\Phi = \{F \in I^X : A \subseteq F$ for some $A \in$ B$\}$ denoted by $\langle$ B $\rangle$. A collection B of subsets of Φ is a base for Φ if each A$\in \Phi$ there is B$\in$ B such that B $\subseteq$ A

Definition 2.2.23 [56]: Let Φ ,$\Psi \in$ FL(I^X) (or FLB(I^X). Then we say that Φ is finer than Ψ, written as $\Psi \subseteq \Phi$, if $\forall A \in \Phi, \exists B \in \Psi$ such that $B \subseteq A$.

Deinition 2.2.24 [56]: Let X be a nonempty set. Then

(i) A fuzzy filter Φ on is called a maximal fuzzy fliter on X iff Φ is finer than every fuzzy filter comparable with it.

(ii) A fuzzy base B on X is called a maximal fuzzy filter base on X if it is base for a maximal fuzzy filter on X.

(iii) A subfamily ξ of fuzzy filter Φ on X is said to be subbase for Φif the family of all finite intersection of members of ξ is a base for Φ. We say that ξ generates Φ.

Result 2.2.25 [85]: (1) Let Φ_1 Φ_2 be any fuzzy filter base on X. then the family

$\Phi_1 \cup \Phi_2 = \{F_1 \cup F_2 : F_1 \in \Phi_1, F_2 \in \Phi_2\}$ is an fuzzy filter base on X.

(2) If $F_1 \cap F_2 \neq \emptyset$ for each $F_1 \in \Phi_1$ and each $F_2 \in \Phi_2$, then $\Phi_1 \cap \Phi_2 = \{ F_1 \cap F_2 : F_1 \in \Phi_1 , F_2 \in \Phi_2 \}$ is a fuzzy filter base on X.

(3) A nonempty family $B \subset I^X$ is an fuzzy filter base iff for any finite collection $\{ B_i \}_1^n$ from B, $\bigcap_{i=1}^{n} B_i \neq \emptyset$

Result 2.2.26 [85]: Let $f : X \to Y$ be a mapping.

(1) If Φ is a fuzzy filter base on X, then so is $f(\Phi) = \{ f(A) : A \in \Phi \}$ on Y.

(2) If Φ is a fuzzy filter base on Y and f is onto, then $f^{-1}(\Phi) = \{ f^{-1}(A) : A \in \Phi \}$ is a fuzzy filter base on X.

Definition 2.2.27 [37]: A filter base Φ is said to be convergent to a fuzzy point P_x^λ, denoted by $\Phi \to P_x^\lambda$, iff every open Q-neighbourhood of P_x^λ contains a member of Φ and $P_x^\lambda \in Cl(A)$ for every $A \in \Phi$ .The limit of a fuzzy filter base Φ is defined by

$\lim(\Phi) = \cup \{ P_x^\lambda \in S(X) : \Phi \to P_x^\lambda$

Definition 2.2.28 [85]: Let (X,T) be a fuzzy topological and Φ a fuzzy filter base on X. A fuzzy point P_x^λ is said to be cluster point (or accumulation point) of Φ if every open Q-neighbourhood U of P_x^λ and for all $F \in \Phi$, we have $F\, q U$.

Result 2.2.29 [37]: A fuzzy point P_x^λ $(0 < \lambda \leq 1)$ in a fts (X,T) is a cluster point of a filter base Φ iff $P_x^\lambda \in Cl(F)$, for each $F \in \Phi$.

Result 2.2.30 [78]: Let (X, T) be a fts and Φ a fuzzy filter (or Fuzzy filter base).Then

(i) $N^Q(P_x^\lambda)$ is a fuzzy filter (filter base) on X and $N^Q(P_x^\lambda) \to P_x^\lambda$

(ii) If P_x^λ is a cluster point of a fuzzy filterbase Φ on X and U is a Q-nbd. of P_x^λ, then $\Psi = \{ F \cap U : F \in \Phi \}$ is finer than Φ and $\Psi \to P_x^\lambda$

(iii) Let A be a nonempty fuzzy set. If $\Phi \to P_x^{\lambda}$ and there exists F$\in \Phi$ such that F $\subseteq$ A, then $p_X^{\lambda} \in cl(A)$

Definition 2.2.31 [6]: A fuzzy set A in an fts X is called a fuzzy regular open set if $\operatorname{int}(cl(A)) = A$. The complements of regularly open sets are called fuzzy regularly closed.

Definition 2.2.32: A fuzzy point p_x^{λ} is said to be a fuzzy δ-cluster [38] (θ-cluster [70]) point of a fuzzy set A in an fts (X, T) if every fuzzy regular open Q-neighbourhood U (respectively the closure of every fuzzy open Q-neighbourhood U) of p_x^{λ} is quasi-coincident with A. The union of all fuzzy δ-cluster (respectively θ-cluster) points of A is called the fuzzy δ-closure (resp. θ-closure) of A and it is denoted by $cl_{\delta}(A)$ (resp. $cl_{\theta}(A)$). A fuzzy set A is fuzzy δ-closed if $cl_{\delta}(A)$= A(resp $cl_{\theta}(A)$ = A) and the complement of such a fuzzy set is called fuzzy δ-open (resp. θ-open).

Definition 2.2.33: The union of all fuzzy θ-open (resp. δ-open) sets contained in a fuzzy set A in an fts (X, T) is called the fuzzy θ-interior [24] (resp. δ-interior [62]) of A, to be denoted by $\operatorname{int}_{\theta}(A)$ (resp. $\operatorname{int}_{\delta}(A)$).

Definition 2.2.34 [64]: An fts X is said to be:

(a) fuzzy regular (semi-regular) iff for each fuzzy point p_x^{λ} in X and each open Q-neighborhood U of p_x^{λ}, ther exists an open Q-neighbourhood V of p_x^{λ} such that $ClV \subseteq U$ (resp. $Int(ClV) \subseteq U$).

(b) fuzzy almost regular iff for each fuzzy point p_x^{λ} in X and each regularly open Q-neighborhood U of p_x^{λ}, there exists a regularly open Q-neighbourhood V of p_x^{λ} such that $ClV \subseteq U$

Result 2.2.35 [70]: An fts X is:

(a) fuzzy regular iff, $cl_{\theta}(A) = clA$, for any fuzzy set A in X

(b) fuzzy semi-regular iff $cl_{\delta}(A) = clA$,for any fuzzy set A in X

Definition 2.2.36 [103]: let f be a function from a set X into a set Y, and let A, B be fuzy sets in X and Y respectively. Then $f(A)$ is a fuzzy set in Y given by:

$$f(A)(y) = \sup\{A(z) : z \in f^{-1}(y)\}, \quad \text{if } f^{-1}(y) \neq \phi$$

$$= 0 \qquad \text{if } f^{-1}(y) = \phi$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by

$f^{-1}(B)(x) = B(f(x))$, for each $x \in X$

Result 2.2.37[21]: let f be a function from a set X into a set Y. Then the followings hold:

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$, where A_1 and A_2 are fuzzy sets in X.

(b) $B_1 \subseteq B_2 \Rightarrow f(B_1) \subseteq f(B_2)$, where B_1 and B_2 are fuzzy sets in Y.

(c) $B \supseteq f(f^{-1}(B))$, for any fuzzy set B in Y.

(d) $A \subseteq f^{-1}(f(B))$, for any fuzzy set A in X.

(e) $f^{-1}(B^c) = (f^{-1}(B))^c$

Result 2.2.38 [6, 93]: let f be a function from a set x into a set Y.

(a) If $A_1, A_2 \in I^X$, then $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

(b) if $A \in I^X$ and $B \in I^Y$, then $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

(c) If $A, B \in I^X$ such that AqB , then $f(A)qf(B)$.

(d) If $A, B \in I^Y$ such that $A\overline{q}B$, then $f^{-1}(A)qf^{-1}(B)$.

Definition 2.2.39: let f be a function from a fts (X, T) into a fts

(Y, T') Then f is said to be:

(i) fuzzy continuous [21] if for each $V \in T'$, $f^{-1}(V) \in T$.

(ii) fuzzy open (fuzzy closed)[21] if for each fuzzy open(resp.fuzzy closed) set A in X , $f(A)$ is fuzzy open (resp. fuzzy closed) in Y.

(iii) fuzzy θ-closed [20] if for each fuzzy closed set A in X , $f(A)$ is fuzzy θ-closed set in Y.

(iv) fuzzy homeomorphism [21] if f is bijective and both f and f^{-1} are fuzzy continuous.

Proposition 2.2.40 [59]: let f be a open function from a fts (X, T) into a fts (Y, T'). Then the followings hold:

(1) $f(\text{int}\, A) \subseteq \text{int}(f(A))$ for each fuzzy set A in X.

(2) $f^{-1}(clB) \subseteq cl(f^{-1}(B))$ for each fuzzy set B in Y

(3) $\text{int}(f^{-1}(B)) \subseteq f^{-1}(\text{int}\, B)$ for each fuzzy set B in Y

Proposition 2.2.41 [58]: let f be a open function from a fts (X, T) into a fts (Y, T'). Then f is closed if and only if $cl(f(A)) \subseteq f(clA)$ for each fuzzy set A in X.

Result 2.2.42 [54]: let f be a function from a fts (X, T) into a fts (Y, T'). Then the following are equivalent:

(1) f is fuzzy continuous

(2) for each fuzzy point p_x^λ in X and each neighbourhood V of $f(p_x^\lambda)$ in Y , there exists a neighbourhood U of p_x^λ such that $f(U) \subseteq V$

(3) For each fuzzy point p_x^λ in X and each open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(U) \subseteq V$.

(4) For any fuzzy set A in X $f(clA) \subseteq clf(A)$.

(5) For any fuzzy set B in Y, $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Definition 2.2.43: A function $f:(X,T) \to (Y,T')$ is said to be:

(a) fuzzy δ-continuous [38] iff for each fuzzy point p_x^λ in X and each regularly open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists a regularly open Q-neighbourhood U of p_x^λ such that $f(U) \subseteq V$.

(b) fuzzy θ-continuous [64] iff for each fuzzy point p_x^λ in X and each open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(clU) \subseteq clV$.

(c) fuzzy weakly δ-continuous [64] iff for each fuzzy point p_x^λ in X and each open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(\operatorname{int} clU) \subseteq clV$

(d) fuzzy weakly θ-continuous [64] iff for each fuzzy point p_x^λ in X and each open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(U) \subseteq clV$

(e) fuzzy almost strongly θ-continuous [64] iff for each fuzzy point p_x^λ in X and each open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(clU) \subseteq \operatorname{int} clV$

(f) fuzzy almost continuous [38] iff for each fuzzy point p_x^λ in X and each regularly open Q-neighbourhood V of $f(p_x^\lambda)$ in Y , there exists an open Q-neighbourhood U of p_x^λ such that $f(U) \subseteq V$

Definition 2.2.44: An fts X is said to be:

(a) fuzzy compact[21] if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there is a finte subset Λ_0 of Λ such that $\bigcup\{U_\alpha : \alpha \in \Lambda_0\} = 1_X$,

(b) fuzzy almost compact if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there is a finte subset Λ_0 of Λ such that $\bigcup\{clU_\alpha : \alpha \in \Lambda_0\} = 1_X$,

(c) fuzzy nearly compact if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there is a finte subset Λ_0 of Λ such that $\bigcup\{IntClU_\alpha : \alpha \in \Lambda_0\} = 1_X$,

Definition 2.2.45: Let $f : X \to Y$ be a fuzzy function. Then graph $g : X \to X \times Y$ of f is defined by $g(p_x^\lambda) = (p_x^\lambda, f(p_x^\lambda))$ for every $p_x^\lambda \in S(X)$.

Definition 2.2.46: Let G_f be graph of a function $f : X \to Y$. Then the inverse of a fuzzy point is defined by $f^{-1}(p_y^r) = \{p_x^\lambda \in S(X) : G_f \neq 0_X\}$