CHAPTER-1

INTRODUCTION AND BACKGROUND

1.1 Fuzzy set theory:

The concept of Fuzzy sets introduced by the American Cybemeticist L. A. Zadeh, started a revolution in every branch of knowledge and particularly in every branch of mathematics. Zadeh introduced the fuzzy set theory [103] in 1965 in order to study the control problem of complicated systems and dealings with fuzzy information. This theory described Fuzziness for the first time. Fuzziness is a kind of uncertainty and uncertainty of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol belongs. Since 16th century probability theory has been studing a kind of uncertainty randomness i. e. the uncertainty of the occurrence of an event. But in this case the event itself is completely certain, the only uncertain thing is whether the event will occur or not. Fuzzy sets are generalization of sets - the basic mathematical object on which modern mathematics is based in set theory, as Cantor defined and Zermelo and Fraenkel axiomatized, an obect is either a number set or not. In fuzzy set theory this condition was relaxed by Lotfi Zadeh and an object is given a degree of membership in a set as nos. between 0 and 1. For example, the degree of membership of a person in the set of rich people is more flexible than a simple yes or no answer and can be a real number such as 0.65. Fuzzy set theory provides us with a frame work which is wider than that of classical set theory. Various mathematical structure whose features emphasize the effects of ordered structure, can be developed on the theory. Fuzzy topology is one such branch combining ordered structure with topological structure. Using fuzzy sets, C. L. Chang defined fuzzy topological space in 1968 for the first time. In his paper Chang confined his attention to the more basic concepts such as open set, closed set, neighbourhood, interior set, continuity and compactness. He followed the definitions, theorems and proofs given by Kelley. Also, Kelly followed the notations and terminology for fuzzy sets as given by Zadeh. His work set a path ablaze for many researchers to define fuzzy topological spaces in several other ways in an attempt of either generalizing the notion or to remove several pathologies existent in the prevailing notions. In 1976, Lowen [55] suggested that constant fuzzy sets are to be included in a fuzzy topology. As a consequence, one more axiom was added to Chang's fuzzy topology. This gave rise to a new school of thought with new orientations and methodologies. In 1974, C. K. Wong [100] introduced the concept of 'fuzzy point belongs to a fuzzy set'. Later the same concept was defined in different ways by Piu and Liu [27], Srivastava, Lal and Srivastava [94]. Pu and Li establish a systematic description of fuzzy point which is also followed in the recent work by most of the mathematicians. After that, in 1980, Pu and Liu [83, 84] introduced the concepts of quasi-coincidence and q-neighbourhoods which again opened up a new avenue by providing with a new fuzzy methodology by which extensions of different aspects in fuzzy setting can be carriedout very interstingly and effiectively. This thesis will make an extensive use of the notions of q-neighbourhood and q-coincidence due to Pu and Liu [83, 84] and the notion of a fuzzy point as introduced in [83, 84, 94].

1.2 Operators on Fuzzy Topological Spaces:

It is well known that in literature there can be found several types of operators like semiclosure [67], δ-closure [94], θ-closure [94], α-closure [74], Preclosure [61] and δ-preclosure [86]. These operators urged to make the corresponding open-like sets namely semi-open, δ-open, θ-open, α-open, β-open, preopen, δ-preopen sets and so on. Applying these new operators and open-like sets, new areas like semi-compactness, near-compactness, α-compactness, β-compactness, δ-continuity, θ-continuity, α-continuity, θ-connectedness, δ-connectedness and various similar concepts have been introduced and studied. Similar attempts have been made to extend these concepts in fuzzy topological spaces. In 1981, Azad [6] introduced and studied fuzzy semiclosure operator and also semiopen (semi-closed) sets. In the same spirit, A. S. Bin Shahna introduced fuzzy α-open (α-closed) sets and fuzzy Preopen (preclosed) sets and then study fuzzy strongly semi-continuous and fuzzy precontinuous mappings. In 1983, Monsef [30] introduced the concepts of β-closure operator and study β-open sets and β-continuous function in general topology and Fath Alla in [34] introduced these concepts in fuzzy setting. After that the notions of β-compactness and β-closed spaces are studied in [5], [42]. It is well-known that the concepts of θ-closure and δ-closure operator of Velico [96] are useful tools in standard topology in the study of H-closed spaces, H-closed extensions etc. Due to varied applicabilities of these operators in formulating various important set toplogical concepts, it is natural to try for their extension to fuzzy topological spaces. With this motivation in mind the concept of δ-closure operator in a fuzzy topological spces was introduced by Ganguly and Saha [22] and made a study of fuzzy δ-continuity, fuzzy δ-connected sets, fuzzy δ-convergent, fuzzy δ-accumulation, fuzzy δ-closed graphs etc. In the same spirit, Mukherjee and Sinha [70] extended the notion of θ-closure operator of Velico [94] in set topological space into a fuzzy topological space and this has opened up a new direction to investigate new concepts in fuzzy topological spaces.

1.3 Functions between Fuzzy Topological Spaces:

Functions and in particular continuous functions stand among the most important and most fertile area in Mathematical Science. So the notion of continuity is an important concept in general topology and fuzzy topology as well as in almost all branches of mathematics. Different types of generalizations of fuzzy continuous functions were introduced and studied by various authors in the recent development of fuzzy topology. Fuzzy topological spaces were first introduced in literature by Chang [21] who studied a number of the basic concepts including fuzzy continuous maps. Then wong [99] introduced fuzzy open and closed mappings .With the help of the notions of a fuzzy point, neighborhood structure of a fuzzy point [62], q-neighborhood of a fuzzy point and quasi-coincidence [84], a study of fuzzy continuity, fuzzy almost continuity and fuzzy weakly continuity has been carried out in [2, 1] as in general topology. Continuing the workdone by the authors in [1], Naseem Ajmal and B. K. Tyagi obtained some more properties of fuzzy almost continuous functions and study the relationship of fuzzy almost continuous mappings with compactness, regularity, almost regularity and normality. In fuzzy topology, Azad [6] was the first who carried forward these ideas to introduce some weaker forms of fuzzy continuity in fuzzy topological spaces. He observed that fuzzy continuity implies fuzzy almost continuity and it again implies fuzzy weakly continuity. M.K. Singal and Niti Rajvananshi [92] proved some properties of fuzzy almost continuous mapping by defining fuzzy almost open mapping. Azad's contribution of fuzzy semi-continuity, almost continuity, weak continuity by semi-open, semi-closed, regular open and closed sets helped Ganguly and Saha, Yalvac [102] and Ghosh to study further in this direction.After that ,there exists several kinds of functions between fuzzy topological spaces. In this context the work done by Arya and Singal [5] and Mukherjee and Sinha [63, 64, 65, 41] is quite appreciable. Takashi Noiri introduced various types of functions between topological spaces and introduced δ-continuity [50], and fuzzy version was introduced by Saha [75]. Fuzzy δ-continuity was also studied by EI-Monsef [29] and Ghosh [40] introduced fuzzy δ-homeomorphism and fuzzy δ-neighbourhoods. Anjana Bhattacharyya and M.N. Mukherijee [14] introduced a generalized class of fuzzy open sets and called them fuzzy δ-preopen. They have studied this type of fuzzy sets to some extent along with the associated notions like fuzzy δ-preclosure, δ-preclosed sets etc. Raychoudhuri and Mukherjee [86] defined δ-almost continuity and δ-preopen sets and Raychoudhuri [87] introduced δ^*-almost continuity, the fuzzy versions was carried out by Anjana Bhattacharyya and M.N. Mukherjee [15]. They have also given a number of characterizations of these two types of functions .Further, worked upon inter relationships and also relations with regard to a neighbouring concept viz. fuzzy almost continuity due to Mukherjee and sinha[65]. They also established the equivalence of the concepts of fuzzy δ-almost continuity and δ^*-almost continuity of a function when the codomain space of the function happens to fuzzy δ-preregular. Similar concept, like δ-continuity and θ-continuity generalized to fuzzy topology by Mukherjee and Sinha [64] and in a subsequent paper [64] deal with two more allied types of functions viz fuzzy almost continuity and fuzzy weak continuity. Among the various types of maps mentioned in this literature, some relatively recent ones are fuzzy α-continuity due to Fath Alla [33] and β-continuity, β-open and β-closed maps due to Mashhour. Among several fuzzy continuous functions, another one that is of great importance is fuzzy c-continuous functions introduced by S.Dang, A.Behera and S.Nanda [25]. In year 1994, the concept of c-continuity of a function in general topology was introduced by Gentry and Hoyle [39]. Long and Hendrix used the concept of c-continuity to investigate some properties of fuzzy T_1-spaces, fuzzy normal spaces and fuzzy Hausdorff spaces are also established. In 1984 D. A. Rose [58] defined weakely open functions in general topological spaces. In 1997, J. H. Park, Y. B. Prk and J. S. Park [82] extend this concept to fuzzy topological spaces. Caldas, Navalagi and Saraf [19] had introduced and studied the concept of fuzzy weakly preopen mappings. A fuzzy weakly pre-open mapping is weaker than fuzzy preopen function introduced by Shahna [91] and fuzzy almost open function introduced by Nanda [72]. They observed that fuzzy preopeness implies fuzzy weakly preopenness but not conversely. Again they derived a condition under which the converse is also true.In 2000, Caldas, Navalagi and Saraf introduced two new classes of functions called fuzzy weakly semi-preopenfunctions and fuzzy weakly semi-preclosed functions. In another paper [20], Caldas, Navalagi and Saraf introduced and discussed the concept of fuzzy θ-closed function and weakly θ-closed function. Also they obtained several characterizations of these function comparing with some already existing known functions.

1.4 Gamma-operations in Topological Space:

An operation γ on the topology T is a mapping from T into the power set P(X) such that V $\subseteq$ γ(V) for each V$\in$T where $\gamma(U)$ denotes the value of γ. The study of this concept was initiated by S.Kasahara [53]. S. Kasahara unified several known characterizations of compact space, nearly compact spaces, and H-closed spaces by introducing a certain operation on a topology. After kasahara, D. S. Jankovic [46] defined the concept of operation closure and investigated some properties of function with operation-closed graphs. Moreover, H. Ogata [79] defined the notion γ-open sets and introduced some new separation axioms. In 1992, F. U. Rehman and B. Ahmad [89] defined and investigated several properties of γ-interior, γ-exterior, γ-closure and γ-boundary points in Topological Spaces and studied the characterizations of (γ, β)-continuous mappings initiated by H. Ogata [79, 88]. After that, for two operations on T, some bioperation-open sets like (γ,γ') , $[\gamma,\gamma']$-open sets and bioperation-separation axioms were defined in [71, 80, 95]. In 2003, B. Ahmad and S. Hussain [11, 12] continued studying the properties of γ-operations on topological spaces. They defined γ-nbd, γ-nbd base at x, γ-closed nbd, γ-limit point, γ-isolated point, γ-convergent point and γ*-regular space and discussed their several properties. They further established the properties of (γ, β)-continuous, (γ, β)- open functions and γ-T_2 spaces. Recently B. Ahamad and S. Hussian contributed many papers in terms of γ-operations. In papers [13, 14, 15, 16, 17], they introduced and studied different topological concepts like γ*-regular, γ0-compact, γ-locally compact and γ-normal spaces, γ-connected spaces etc. and investigated more properties of these notions.

1.5 Some Operations in Fuzzy Topological Spaces:

In 1989, an operation on a topological (X, T) spaces was defined by S. Kasahara [103] as a mapping γ from T into P(X) such that A $\subseteq \gamma(A)$ for all $A \in T$. Then Abd EI-Monsef et. al. extended Kasahara's operation by introducing an operation φ on the power set P(X) of X endowed with a topology T such that Int(A) $\subseteq \varphi$(A) for all A$\in$ P(X). In 1993, Kerre, Naugh and Kandil [54] extended this concept to the class of all fuzzy sets on X endowed with a Chang fuzzy Topology T. They introduced the class of all φ-open fuzzy sets that generalized the classes of all open, semi-open [6], pre-open [34], semi-pre-open and feebly open fuzzy sets. For two operations φ_1, φ_2, they defined the concepts of $\varphi_{1,2}$-closure($\varphi_{1,2}$-interior) of fuzzy sets that generalized fuzzy closure [84], fuzzy θ-closure [70], fuzzy δ-closure [38], fuzzy semi-closure [40], fuzzy semi-θ-closure and fuzzy semi-δ-closure. Also they investigated some types of fuzzy separation axioms due to Pu and Liu [84], Azad [6] and Kandil and EI-Etriby [51]. Using operation on I^X, Ekici (2003) [32] studied some characterizations and properties of fuzzy continuous functions and its weaker and stronger forms including fuzzy weakly continuous, fuzzy θ-continuous, fuzzy strongly θ-continuous, fuzzy almost strongly θ-continuous, fuzzy weakly θ-continuous, fuzzy almost continuous, fuzzy super continuous, fuzzy δ-continuous. Furthermore, Ekicki (2004) [31] presented the same work for fuzzy multifunctions. By operation, Ekici investigated some characterizations and some properties of fuzzy lower and upper continuous multifunctions and its weaker and stronger forms including fuzzy lower and upper weakly continuous, fuzzy lower and upper θ-continuous, fuzzy lower and upper strongly θ-continuous, fuzzy lower and upper almost strongnly θ-continuous,fuzzy lower and upper weakly θ-continuous, fuzzy, lower and upper almost continuous, fuzzy lower and upper super continuous, fuzzy lower and upper δ-continuous. Recently Babitha and Sitrarasu (2010) [17] defined an operation γ in Generalized fuzzy topological spaces and studied γ -fuzzy open set, γ-fuzzy interior, γ-fuzzy closure. Also they investigated some properties of these notions.