CHAPTER-6
BIOPERATION-OPEN SETS AND BIOPERATION-CONTINUOUS FUNCTIONS IN FUZZY TOPOLOGICAL SPACE

6.1 Introduction:

In this chapter, we generalize the notion of operation-open sets in the sense of chapter-3 to bioperations and define bioperation-closure and bioperation-generalized closed sets. We then study the concepts of fuzzy bioperation-continuities and bioperation-separation axioms. Several properties and characterizations of these notions are also investigated. Throughout this chapter, γ and γ' are two operations on fuzzy topology T. The results contained in this chapter are communicated in the form of papers [45],[49] for publication.

6.2: Fuzzy (γ,γ')-open sets and its properties:

In this section we have defined the notion of fuzzy (γ,γ')-open sets and investigate the relation between fuzzy (γ,γ')-open sets and fuzzy γ-open sets [chapter-3]

Definition 6.2.1: A fuzzy subset A of (X, T) is called a fuzzy (γ,γ')-open set if for each p_x^λ q A, there exist open q-neighborhood U and V of p_x^λ such that $\gamma(U)\cup\gamma'(V)\subseteq A$

Theorem 6.2.2: Let A be a fuzzy subset of (X, T).

(i) A is fuzzy (γ,γ')-open if and only if A is fuzzy γ-open and fuzzy γ'-open.

(ii) If A is fuzzy (γ,γ')-open, then A is open

(iii) If A_j is fuzzy (γ,γ')-open for every j∈ J, then $\cup\{A_j \mid j\in J\}$ is fuzzy (γ,γ')-open.

(iv) The following statements are equivalent:

(a) A is fuzzy (γ,γ)-open.

(b) A is fuzzy γ-open

Proof: (i) (Necessity) Let p_x^λ q A. Then there exists open q-neighborhoods U and V of p_x^λ such that $\gamma(U)\cup\gamma'(V)\subseteq$ A. Accordingly max{ γ(U)(x) , γ'(V)(x) } ≤ A(x) for all $x\in X$ and so γ(U)(x) ≤ A(x) and γ'(V)(x) ≤ A(x). Therefore $\gamma(U)\subseteq A$ and $\gamma'(V)\subseteq A$. Hence A is γ-open and γ'-open

(Sufficiency): Let p_x^λ q A. Since A is fuzzy γ-open and fuzzy γ'-open, there exists open q-neighborhoods U and V of p_x^λ such that $\gamma(U)\subseteq A$ and $\gamma'(V)\subseteq A$. Then we obtain $\gamma(U)\cup\gamma'(V)\subseteq A$ and so A is fuzzy (γ,γ')-open

(ii) Let A be (γ,γ')-open set. Since $T_\gamma\subseteq T$ and A is γ-open by (i), A is open.

(iii) Let B = $\bigcup\{ A_j \mid j\in J \}$ and p_x^λ q B. Then there exists some $A_j \in T$ such that p_x^λ q A_j. Since A_j is (γ,γ')-open, there exists open q-neighborhoods U and V of p_x^λ such that $\gamma(U)\cup\gamma'(V)\subseteq A_j$. Therefore, ($\gamma$(U) $\cup$ γ'(V))(x) ≤ A_j(x) for all $x\in X$ and so (γ(U) $\cup$ γ'(V))(x) ≤ sup{A_j(x) | j∈ J }. This implies $\gamma(U)\cup\gamma'(V)\subseteq$ B and hence B is (γ,γ')-open.

(iv) (a) ⇔ (b) follows if $\gamma = \gamma'$ in (i).

Corollary 6.2.3: $T_{(\gamma,\gamma')}$ denotes the set of all fuzzy (γ,γ')-open sets of (X,T). Then from the theorem 6.2.2, we can obtain the following relation

$$T_{(\gamma,\gamma')} = T_\gamma \cap T_{\gamma'} \subseteq T .$$

Definition 6.2.4: A fuzzy topological space (X,T) is said to be fuzzy (γ,γ')-regular space if for each fuzzy point p_x^λ of X and every open q-neighborhood U of p_x^λ there exist open q-neighborhoods W and S of p_x^λ such that γ(W) $\cup$ γ'(S) $\subseteq$ U.

Theorem 6.2.5: Let (X, T) be fuzzy topological space.Then

(i) (X, T) is fuzzy (γ, γ')-regular if and only if $T_{(\gamma,\gamma')} = T$ holds.

(ii) (X, T) is fuzzy (γ, γ')-regular if and only it is fuzzy γ-regular and fuzzy γ'-regular.

(iii) The following statements are equivalent:

(a) (X, T) is fuzzy (γ, γ)-regular.

(b) (X, T) is fuzzy γ-regular.

Proof: (i) (Necessity) Since $T_{(\gamma,\gamma')} \subseteq T$, it is sufficient to prove $T \subseteq T_{(\gamma,\gamma')}$. Let A $\in$ T and p_x^λ q A. Then A(x) > 1 - λ and so A is open q-neighborhood of p_x^λ. Since (X, T) is fuzzy (γ, γ')-regular , there exists open q-neighborhoods W and S of p_x^λ such that γ(W) $\cup$ γ'(S) $\subseteq$ A. This shows that A is fuzzy (γ, γ')-open set.

(sufficiency) Let p_x^λ be a fuzzy point in X and let V be open q-neighborhood of p_x^λ. Since $T_{(\gamma,\gamma')} = T$, V is fuzzy (γ, γ')-open set. Therefore there exists open Q-neighborhoods W and S of p_x^λ such that γ(W) $\cup$ γ'(S) $\subseteq$ V. This shows (X,T) is fuzzy (γ, γ')-regular .

(ii) By using (i) and 6.2.3, (X,T) is fuzzy (γ, γ')-regular if and only if $T_{(\gamma,\gamma')} = T_\gamma \cap T_{\gamma'} = T$ .That is, (X, T) is fuzzy (γ, γ')-regular if and only $T = T_\gamma = T_{\gamma'}$.

By using theorem 3.2.15, we can obtain that (X,T) is fuzzy (γ, γ')-regular if and only it is fuzzy γ-regular and fuzzy γ'-regular.

(iii) It is shown by setting $\gamma = \gamma'$ in (i) and using theorem 3.2.15

Proposition 6.2.6: Let γ and γ' be fuzzy regular operations.

(i) If A and B are fuzzy (γ, γ')-open sets, then A $\cap$ B is (γ, γ')-open.

(ii) $T_{(\gamma,\gamma')}$ is a fuzzy topology on X.

Proof: (i) Let p_x^λ q (A $\cap$ B). Then p_x^λ q A and p_x^λ q B. By theorem 6.2.2, A and B are both fuzzy γ-open and fuzzy γ'-open. Then there exist open Q-neighborhoods U, V, W, and S of p_x^λ such that γ(U)$\subseteq$A, γ'(W)$\subseteq$A and γ(V)$\subseteq$ B and γ'(S) $\subseteq$ B.

Now (γ(U)$\cap$ γ(V))(x) = min { γ(U)(x), γ(V)(x)}

$\leq$ min { A(x), B(x) }

= (A$\cap$B)(x)

and (γ'(W)$\cap$ γ'(S)) (x) = min { γ'(W)(x), γ'(S)(x) }

$\leq$ min { A(x), B(x) }

= (A$\cap$B)(x)

Therefore

((γ(U)$\cap$ γ(V))$\cup$((γ'(W)$\cap$ γ'(S)))(x) = max { (γ(U)$\cap$ γ(V))(x), (γ'(W)$\cap$ γ'(S)) (x)

$\leq$ max { (A$\cap$B)(x), (A$\cap$B)(x) }

= (A$\cap$B)(x)

By using regularity of γ and γ', there exist open Q-neighborhoods E and F of p_x^λ such that γ(E)$\subseteq$ γ(U)$\cap$ γ(V) and γ'(F)$\subseteq$ γ'(W)$\cap$ γ'(S).

Hence (γ(E)$\cup$ γ'(F))(x) = max { γ(E)(x), γ'(F)(x) }

$\leq$ max { (γ(U)$\cap$ γ(V))(x), (γ'(W)$\cap$ γ'(S))(x) }

$\leq$ max { (A$\cap$B))(x), (A$\cap$B)(x) }

= (A$\cap$B))(x)

So, γ(E) $\cup$ γ'(F)$\subseteq$ A$\cap$B. This implies that A$\cap$B is fuzzy (γ,γ')-open set.

(ii) 0 and 1 are fuzzy (γ,γ')-open sets together with (i) and theorem 6.2.2 (iii) $T_{(\gamma,\gamma')}$ is fuzzy topology on X.

6.3. Fuzzy (γ,γ')-closures and its properties:

In this section we have defined two different types of fuzzy bioperation-closures and investigated relations between them.

Definition 6.3.1: A fuzzy subset A of (X,T) is said to be fuzzy (γ,γ')-closed set if its complement A^c is fuzzy (γ,γ')-open set

Definition 6.3.2: For a fuzzy subset A of (X,T) and $T_{(\gamma,\gamma')}$, $T_{(\gamma,\gamma')}$-Cl(A) denotes the intersection of all (γ,γ')-closed sets containing A i.e.

$$T_{(\gamma,\gamma')}\text{-Cl}(A) = \inf \{ F : A \subseteq F,\ F^c \in T_{(\gamma,\gamma')} \}.$$

The following proposition characterizes $T_{(\gamma,\gamma')}$-Cl(A).

Theorem 6.3.3: (i) For a fuzzy point p_x^λ in X, $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A) if and only V q A for any $V \in T_{(\gamma,\gamma')}$ and p_x^λ q V.

(ii) A is (γ,γ')-closed if and only if $T_{(\gamma,\gamma')}$-Cl(A) = A

Proof: We have $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A) if and only if for every fuzzy (γ,γ')-closed set $F \supseteq A$, $p_x^\lambda \in F$ or $F(x) \geq \lambda$. By taking complement, this fact can be stated as follows: $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A)if and only if for every fuzzy (γ,γ')-open set $B \subseteq A^c$, $B(x) \leq 1-\lambda$.

In other words, $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A) if and only if for every fuzzy (γ,γ')-open set B satisfying $B(x) > 1-\lambda$ and B is not contained in A^c (which implies BqA). Thus we have proved that $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A) if and only if V q A for every fuzzy (γ,γ')-open set V and p_x^λ q V.

(ii) (Necessity) Let A be fuzzy (γ,γ')-closed set. Then by definition 6.3.2 $T_{(\gamma,\gamma')}$-Cl(A) = A.

(Sufficiency) Let $T_{(\gamma,\gamma')}$-Cl(A) = A. We want to prove that A^c is fuzzy (γ,γ')-open set. Let p_x^λ q A^c. Then $p_x^\lambda \notin$ A $= T_{(\gamma,\gamma')}$-Cl(A) and there exists a fuzzy (γ,γ')-open set V and p_x^λ q V such that V is not quasi-concident with A. Therefore $V \subseteq A^c$. Since V is fuzzy (γ,γ')-open set, for p_x^λ q V, there exists open q-neighborhoods W and S of p_x^λ such that γ(W) ∪γ(S) ⊆ V. Hence we have γ(W)∪γ(S) $\subseteq A^c$. This shows that A^c is fuzzy (γ,γ')-open set. That is A is fuzzy (γ,γ')-closed.

Theorem 6.3.4: Let A and B be fuzzy subsets of (X,T).

(i) A ⊆ $T_{(\gamma,\gamma')}$-Cl(A),

(ii) If A ⊆ B, then $T_{(\gamma,\gamma')}$-Cl(A)⊆ $T_{(\gamma,\gamma')}$-Cl(B).

(iii) $T_{(\gamma,\gamma')}$-Cl(A) is fuzzy (γ,γ')-closed set.

***Proof*:** (i) It is obvious from definition 6.3.2.

(ii) Let $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A). Then by theorem 6.3.3 (i), we have V q A for every fuzzy (γ,γ')-open set V and p_x^λ q V. Since A⊆ B, we have V q B. This shows that $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(B) . Thus $T_{(\gamma,\gamma')}\text{-}cl(A) \subseteq T_{(\gamma,\gamma')}\text{-}cl(B)$

(iii). Here we prove that $T_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) = $\tau_{(\gamma,\gamma')}$-Cl(A). Let us put G = $T_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) and H = $T_{(\gamma,\gamma')}$-Cl(A). Let $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) and V be fuzzy γ-open set and p_x^λ q V. Then by theorem 6.3.3 (i), we have V q H. This implies V(x) +H(x) > 1 for some $x \in X$. Let H(x) = r, r∈ [0,1]. Then $p_x^r \in$ H = $T_{(\gamma,\gamma')}$-Cl(A) and V is fuzzy (γ,γ')-open set and p_x^r q V. Hence by theorem 6.3.3 (i) we get V q A. This shows that $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A).

Again, let $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl(A).Then by (i), $p_x^\lambda \in T_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A))). Thus we have shown that $p_x^\lambda \in \tau_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) $\Leftrightarrow$ $p_x^\lambda \in (T_{(\gamma,\gamma')}$-Cl(A)). Hence

$T_{(\gamma,\gamma')}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) = $T_{(\gamma,\gamma')}$-Cl(A) and by theorem 6.3.3 (ii) $T_{(\gamma,\gamma')}$-Cl(A) is fuzzy (γ, γ')-closed set.

We introduce the following definition of Cl$_{(\gamma,\gamma')}$(A).

Definition 6.3.5: Let $p_x^\lambda \in S(X)$ and $A \in I^X$. Then the fuzzy (γ, γ')-closure of A, denoted by Cl$_{(\gamma,\gamma')}$(A), given by :

$p_x^\lambda \in Cl_{(\gamma,\gamma')}(A)$ iff (γ(V)∪γ(W)) q A for each open q-neighborhoods V and W of p_x^λ.

Theorem 6.3.6: Let A be a fuzzy subset of (X,T). Then

Cl$_{(\gamma,\gamma')}$(A) = Cl$_\gamma$(A) ∪ Cl$_{\gamma'}$(A) holds, where Cl$_\gamma$(A) and Cl$_{\gamma'}$(A) are γ-closure and γ'-closure of A respectively .

Proof: We have

$p_x^\lambda \notin$ Cl$_{(\gamma,\gamma')}$(A).

$\Leftrightarrow$ There exist open q-neighborhoods V and W of p_x^λ such that

γ(V) ∪γ(W) is not quasi-concident with A.

$\Leftrightarrow$ There exist open q-neighborhoods V and W of p_x^λ such that

(γ(V) ∪γ(W))(x) + A(x) ≤ 1.

$\Leftrightarrow$ There exist open q-neighborhoods V and W of p_x^λ such that

max { γ(V)(x), γ'(W)(x)} + A(x) ≤ 1.

$\Leftrightarrow$ There exist open q-neighborhoods V and W of p_x^λ such that

γ(V)(x) + A(x) ≤ 1 and γ'(W) + A(x) ≤ 1.

$\Leftrightarrow$ $p_x^\lambda \notin$ Cl$_\gamma$(A) and $p_x^\lambda \notin$ Cl$_{\gamma'}$(A).

$\Leftrightarrow$ $Cl_{\gamma}(A)(x) < \lambda$ and $Cl_{\gamma'}(A)(x) < \lambda$.

$\Leftrightarrow$ Max { $Cl_{\gamma}(A)(x)$, $Cl_{\gamma'}(A)(x)$} $< \lambda$.

$\Leftrightarrow$ $p_x^{\lambda} \notin Cl_{\gamma}(A) \cup Cl_{\gamma'}(A)$.

Hence $Cl_{(\gamma,\gamma')}(A) = Cl_{\gamma}(A) \cup Cl_{\gamma'}(A)$.

Theorem 6.3.7: For a fuzzy subset A of (X,T) the following properties hold.

(i) $A \subseteq Cl(A) \subseteq Cl_{(\gamma,\gamma')}(A) \subseteq T_{(\gamma,\gamma')}\text{-}Cl(A)$

(ii) If $A \subseteq B$ then $Cl_{(\gamma,\gamma')}(A) \subseteq Cl_{(\gamma,\gamma')}(B)$

Proof: (i). By Theorem 6.3.6 and theorem 3.3.6, it is shown that

$Cl_{(\gamma,\gamma')}(A) = Cl_{\gamma}(A) \cup Cl_{\gamma'}(A) \supseteq Cl(A)$.

Now we show that $Cl_{(\gamma,\gamma')}(A) \subseteq T_{(\gamma,\gamma')}\text{-}Cl(A)$.

Let $p_x^{\lambda} \notin T_{(\gamma,\gamma')}\text{-}Cl(A)$.Then there exists an fuzzy (γ,γ')-open V such that p_x^{λ} q V and V is not quasi-concident with A. Since V is fuzzy (γ,γ')-open set, so there exists open Q-neighborhoods W and S of p_x^{λ} such that $\gamma(W) \cup \gamma'(S) \subseteq V$. Therefore $\gamma(W) \cup \gamma'(S)$ is not quasi-concident with A. Then we have $p_x^{\lambda} \notin Cl_{(\gamma,\gamma')}(A)$. Hence

$Cl_{\gamma}(A) \subseteq T_{(\gamma,\gamma')}\text{-}Cl(A)$ .Thus we have $A \subseteq Cl(A) \subseteq Cl_{(\gamma,\gamma')}(A) \subseteq T_{(\gamma,\gamma')}\text{-}Cl(A)$.

(ii) Let $p_x^{\lambda} \in Cl_{(\gamma,\gamma')}(A)$. Let W and S be fuzzy open q-neighborhoods of p_x^{λ}. Then we have ($\gamma(W) \cup \gamma(S)$) q A. Since $A \subseteq B$ so we get ($\gamma(W) \cup \gamma(S)$) q B. This shows

$p_x^{\lambda} \in Cl_{(\gamma,\gamma')}$ (B). Hence $Cl_{(\gamma,\gamma')}$ (A) $\subseteq Cl_{(\gamma,\gamma')}$ (B).

Theorem 6.3.8: Let A be a fuzzy subset of (X,T).

i. A is fuzzy (γ,γ')-closed if and only if $Cl_{(\gamma,\gamma')}(A) = A$.

ii. $T_{(\gamma,\gamma')}$-Cl(A)=A if and only if Cl$_{(\gamma,\gamma')}$(A) =A.

iii. A is fuzzy (γ,γ')-open if and only if Cl$_{(\gamma,\gamma')}$(A^c) = A^c.

Proof: (i) (Necessity): we prove that Cl$_{(\gamma,\gamma')}$(A) $\subseteq$ A. Let $p_x^\lambda \notin$ A . Then p_x^λ q A^c. Since A^c is fuzzy (γ,γ')-open, there exists open Q-neighborhoods W and S of p_x^λ such that γ(W)∪γ(S) $\subseteq$ A^c and so γ(W)∪γ(S) is not quasi-concident with A. It shows that $p_x^\lambda \notin$ Cl$_\gamma$(A). Hence Cl$_{(\gamma,\gamma')}$(A) $\subseteq$ A. Again by theorem 6.3.7(i) , we have A $\subseteq$ Cl$_{(\gamma,\gamma')}$(A). Thus Cl$_{(\gamma,\gamma')}$(A) = A.

(Sufficiency): We want to prove that A^c A^c is fuzzy (γ,γ')-open. Let Let p_x^λ q A^c.Then $p_x^\lambda \notin$ A= Cl$_{(\gamma,\gamma')}$(A) and there exists fuzzy open q-neighborhoods W and S of p_x^λ such that γ(W)∪γ(S) is not quasi-concident with A This implies that γ(W)∪γ(S)) $\subseteq$ A^c. Therfore A^c is fuzzy (γ,γ')-open so that A is (γ,γ')-closed.

(ii) It is proved by (i) and theorem 6.3.3 (ii).

(iii) It follows from (i) and definition 6.3.1.

Theorem 6.3.9: For a fuzzy subset A of (X,T), the following properties hold:

(i) If (X,T) is fuzzy (γ,γ')-regular space then Cl(A) = Cl$_{(\gamma,\gamma')}$(A) = $T_{(\gamma,\gamma')}$-Cl(A)

(ii) Cl$_{(\gamma,\gamma')}$(A) is fuzzy closed subset of (X,T).

(iii) $T_{(\gamma,\gamma')}$-Cl(Cl$_{(\gamma,\gamma')}$(A)) = $T_{(\gamma,\gamma')}$-Cl(A) = Cl$_{(\gamma,\gamma')}$ ($T_{(\gamma,\gamma')}$-Cl(A))

Proof: (i) By theorem 6.2.5 (i), we have T = $T_{(\gamma,\gamma')}$ and hence Cl(A)) = $T_{(\gamma,\gamma')}$-Cl(A). By using theorem 6.3.7 (i), it is shown that Cl(A) = Cl$_{(\gamma,\gamma')}$(A) = $T_{(\gamma,\gamma')}$-Cl(A).

(ii) It follows from Theorem 6.3.5 and theorem 3.3.6(iii) that

Cl(Cl$_{(\gamma,\gamma')}$(A)) = Cl(Cl$_\gamma$(A) ∪ Cl$_{\gamma'}$(A))= Cl(Cl$_\gamma$(A))∪ Cl(Cl$_{\gamma'}$(A)) =

Cl$_\gamma$(A) ∪ Cl$_{\gamma'}$(A) = Cl$_{(\gamma,\gamma')}$(A).

(iii) By the theorem 6.3.4(iii) we have $T_{(\gamma,\gamma')}$-Cl(A) is fuzzy γ-closed subset of X. Then by theorem 6.3.8(i) we get $T_{(\gamma,\gamma')}$-Cl(A) = $\mathrm{Cl}_{(\gamma,\gamma')}(T_{(\gamma,\gamma')}$-Cl(A)). Again by theorem 6.3.7 (i) we have A $\subseteq$ $\mathrm{Cl}_{(\gamma,\gamma')}$ (A). Then by theorem 6.3.4(ii) we get

$T_{(\gamma,\gamma')}$-Cl(A) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl($\mathrm{Cl}_{(\gamma,\gamma')}$(A)). Since by theorem 6.3.7(i)

$\mathrm{Cl}_{(\gamma,\gamma')}$(A) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl(A), we obtain that

$\mathrm{Cl}_{(\gamma,\gamma')}$(A) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl(A) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl($\mathrm{Cl}_{(\gamma,\gamma')}$(A)).By using these inclusions and theorem 6.3.4 (ii), we obtain that

$T_{(\gamma,\gamma')}$-Cl ($\mathrm{Cl}_{(\gamma,\gamma')}$(A)) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl($\tau_{(\gamma,\gamma')}$-Cl(A)) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl ($T_{(\gamma,\gamma')}$-Cl($\mathrm{Cl}_{(\gamma,\gamma')}$(A)).

By theorem 6.3.4 (iii) it can be written as

$T_{(\gamma,\gamma')}$-Cl ($\mathrm{Cl}_{(\gamma,\gamma')}$(A)) $\subseteq$ $T_{(\gamma,\gamma')}$-Cl(A) $\subseteq$ ($T_{(\gamma,\gamma')}$-Cl($\mathrm{Cl}_{(\gamma,\gamma')}$(A)).Thus we get

$T_{(\gamma,\gamma')}$-Cl(A)= $T_{(\gamma,\gamma')}$-Cl ($\mathrm{Cl}_{(\gamma,\gamma')}$(A)) and hence

$T_{(\gamma,\gamma')}$-Cl ($\mathrm{Cl}_{(\gamma,\gamma')}$(A)) = $T_{(\gamma,\gamma')}$-Cl(A)= $\mathrm{Cl}_{(\gamma,\gamma')}(T_{(\gamma,\gamma')}$-Cl(A)).

Theorem 6.3.10: Let γ and γ' be fuzzy operations and A a fuzzy subset of (X, T). If T_{γ} = $T_{\gamma'}$ holds, then

(i) $\mathrm{Cl}_{(\gamma,\gamma')}$(A) = $T_{(\gamma,\gamma')}$-Cl(A), and

(ii) $\mathrm{Cl}_{(\gamma,\gamma')}$($\mathrm{Cl}_{(\gamma,\gamma')}$(A)) = $\mathrm{Cl}_{(\gamma,\gamma')}$(A).i.e. $\mathrm{Cl}_{(\gamma,\gamma')}$(A) is fuzzy ($\gamma,\gamma'$)-closed set..

Proof: (i) By (6.2.3), we have $T_{(\gamma,\gamma')}$ = T_{γ} = $T_{\gamma'}$ and hence $\tau_{(\gamma,\gamma')}$-Cl(A) = T_{γ}-Cl(A) = $T_{\gamma'}$-Cl(A). Now using Theorem 6.3.6 , we obtain $T_{(\gamma,\gamma')}$-Cl(A) = T_{γ}-Cl(A)$\cup$ $T_{\gamma'}$-Cl(A) = Cl_{γ}(A) $\cup$ $\mathrm{Cl}_{\gamma'}$(A) = $\mathrm{Cl}_{(\gamma,\gamma')}$(A).

(ii) It is proved by Theorem (i) and Theorem 6.3.9(iii).

6.4 Fuzzy (γ,γ')-separations axioms:

In this section we introduce fuzzy (γ,γ')-g-closed set and fuzzy $(\gamma,\gamma')-T_i\,(i=1,2,\frac{1}{2})$ spaces and obtain some their properties. Throughout this section, γ and γ' be given two fuzzy operations on fuzzy topology T and $X\times X$ the fuzzy product of X and $\Delta(X)=\{(p_x^\lambda,p_x^\lambda):p_x^\lambda\in S(X)\}$.

Definition 6.4.1: A fts (X,T) is called fuzzy (γ,γ')-T_1 iff for each $(p_x^\lambda,p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U and V of p_x^λ and p_y^k respectively such that $p_y^k\overline{q}\gamma(U)$ and $p_x^\lambda\overline{q}\gamma'(V)$

Remark 6.4.2: For given two distinct fuzzy points p_x^λ and p_y^k ,the fuzzy (γ,γ')-T_1-axioms requires that there exists open Q-neighbourhoods U , W of p_x^λ and V, S of p_y^k respectively such that $p_y^k\overline{q}\gamma(U)$ and $p_x^\lambda\overline{q}\gamma'(V)$, and $p_y^k\overline{q}\gamma'(W)$ and $p_x^\lambda\overline{q}\gamma(S)$. Clearly (X,T) is fuzzy (γ,γ)-T_1 if and only if (X,T) is γ-T_1.

Definition 6.4.3: A fts (X,T) is called fuzzy (γ,γ')-T_2 iff for each $(p_x^\lambda,p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U and V of p_x^λ and p_y^k respectively such that $\gamma(U)\overline{q}\gamma'(V)$.

Remark 6.4.4: For given two distinct fuzzy points p_x^λ and p_y^k, the fuzzy (γ,γ')-T_2-axioms requires that there exists open Q-neighbourhoods U, W of p_x^λ and V, S of p_y^k such that $\gamma(U)\overline{q}\gamma'(V)$ and $\gamma'(W)\overline{q}\gamma(S)$. Clearly (X,T) is fuzzy (γ,γ)-T_2 if and only if (X,T) is γ-T_2.

Theorem 6.4.5: A space (X,T) is fuzzy (γ,γ')-T_1 if and only any fuzzy singletons in X is a fuzzy (γ,γ')-closed set.

Proof: (Necessity): Let (X,T) be a fuzzy (γ,γ')-T_1 and $p_x^\lambda \in$ S(X). Since $p_x^\lambda \subseteq cl_{(\gamma,\gamma')}(p_x^\lambda)$, so it is only need to prove $cl_{(\gamma,\gamma')}(p_x^\lambda) \subseteq p_x^\lambda$. Let $p_y^k \notin p_x^\lambda$.Then for each $(p_x^\lambda, p_y^k) \in X \times X - \Delta(X)$, there exists open Q-neighbourhoods U and V of p_x^λ and p_y^k respectively such that $p_y^k \overline{q}\, \gamma(U)$ and $p_x^\lambda \overline{q}\, \gamma'(V)$. Also for each $(p_y^k, p_x^\lambda) \in X \times X - \Delta(X)$, there exists open Q-neighbourhoods W of p_y^k and S of p_x^λ such that $p_y^k \overline{q}\, \gamma'(W)$ and $p_x^\lambda \overline{q}\, \gamma(S)$. Therefore we have $(\gamma(S) \bigcup \gamma'(V))\overline{q} p_x^\lambda$. This means that $p_y^k \notin cl_{(\gamma,\gamma')}(p_x^\lambda)$. Thus $cl_{(\gamma,\gamma')}(p_x^\lambda) \subseteq p_x^\lambda$

(sufficiency): Let $p_x^\lambda, p_y^k \in$ S(X) and $p_x^\lambda \neq p_y^k$. Since p_x^λ and p_y^k are both (γ,γ')-closed set, $cl_{(\gamma,\gamma')}(p_x^\lambda) = p_x^\lambda$ and $cl_{(\gamma,\gamma')}(p_y^k) = p_y^k$. Since $p_x^\lambda \neq p_y^k$, then $p_y^k \notin cl_{(\gamma,\gamma')}(p_x^\lambda)$ and $p_x^\lambda \notin cl_{(\gamma,\gamma')}(p_y^k)$. Therefore, there exists open Q-neighbourhoods U,W of p_x^λ and V,S of p_y^k such that $(\gamma(U) \bigcup \gamma'(W))\overline{q} p_y^k$ and $(\gamma(V) \bigcup \gamma'(S))\overline{q} p_x^\lambda$. This implies such that $p_y^k \overline{q}\, \gamma(U)$ and $p_x^\lambda \overline{q}\, \gamma'(S)$, and $p_x^\lambda \overline{q}\, \gamma(V)$ and $p_y^k \overline{q}\, \gamma'(W)$.Thus for each $(p_x^\lambda, p_y^k) \in X \times X - \Delta(X)$,there exists open Q-neighbourhoods A and B of p_x^λ and p_y^k respectively such that $p_y^k \overline{q}\, \gamma(A)$ and $p_x^\lambda \overline{q}\, \gamma'(B)$. This implies (X,T) is fuzzy (γ,γ')-T_1 spaces.

Theorem 6.4.6: If a space (X,T) is fuzzy (γ,γ')-T_2, then it is fuzzy (γ,γ')-T_1.

Proof: Let (X,T) be a fuzzy (γ,γ')-T_2 space. Then for each $(p_x^\lambda, p_y^k) \in X \times X - \Delta(X)$, there exists open Q-neighbourhoods U and V of p_x^λ and p_y^k respectively such that $\gamma(U)\overline{q}\, \gamma'(V)$.Since $p_x^\lambda q \gamma(U)$, $p_y^k q \gamma(V)$ and $p_x^\lambda \neq p_y^k$, therefore $p_x^\lambda \overline{q}\, \gamma'(V)$ and $p_y^k \overline{q}\, \gamma(U)$. Hence (X,T) is (γ,γ')-T_1.

Definition 6.4.7: Let (X,T) be a fts and γ an operation on T. A fuzzy set $A \in I^X$ is called (γ,γ')-generalized closed ((γ,γ')-g-closed, for short) if $cl_{(\gamma,\gamma')}(A) \subseteq U$ whenever $A \subseteq U$ and U is fuzzy (γ,γ')-open in (X,T).

Theorem 6.4.8: Every fuzzy (γ,γ')-closed set is fuzzy (γ,γ')-g-closed.

Proof: Obvious. The converse is not true as shown by the following example.

Example 6.4.8: Let X = {x,y} and $T = \{X, \varnothing, p_y^{0.7}\}$. Define $\gamma : T \to I^X$ by $\gamma(U) = cl(U) = \gamma'(U)$ for each $U \in T$. Let $A = p_x^{0.5} \cup p_y^{0.6}$. Then A is fuzzy (γ,γ')-g.closed set but not fuzzy (γ,γ')-closed set.

Definition 6.4.9: A space (X,T) is called a fuzzy (γ,γ')-$T_{1/2}$ space if every fuzzy (γ,γ')-g-closed set of (X,T) is fuzzy (γ,γ')-closed

Theorem 6.4.10: For each $p_x^\lambda \in$ S(X), p_x^λ is (γ,γ')-closed or $(p_x^\lambda)^C$ is fuzzy (γ,γ')-g.closed set in (X,T).

Proof: Suppose p_x^λ is not (γ,γ')-closed. Then $(p_x^\lambda)^C$ is fuzzy (γ,γ')-open. Let U be any fuzzy (γ,γ')-open set such that $(p_x^\lambda)^C \subseteq U$. Since $U = X$ is the only fuzzy (γ,γ')-open, $cl_{(\gamma,\gamma')}((p_x^\lambda)^C) \subseteq U$. Therefore $(p_x^\lambda)^C$ is fuzzy (γ,γ')-g.closed set.

6.5. Fuzzy [γ,γ']-open sets and its properties:

In this section, we introduce an altenative fuzzy bioperation-open sets of type [γ,γ'] and investigate relations between it and that of fuzzy (γ,γ')-open sets and fuzzy γ-open sets [chapter-3] are investigated

Definition 6.5.1: A fuzzy subset A of (X, T) will be called a fuzzy [γ, γ']-open set if for each p_x^λ q A, there exists open q-neighborhoods U and V of p_x^λ such that $\gamma(U) \cap \gamma(V) \subseteq$ A.

Theorem 6.5.2: Let A be a fuzzy subset of (X, T).

(i) If A is fuzzy γ-open and B is fuzzy γ'-open then A $\cap$ B is fuzzy [γ, γ']-open.

(ii) If A is fuzzy [γ, γ'] -open, then A is open

(iii) If A_j is fuzzy [γ, γ']-open for every j$\in$ J, then $\bigcup$ {A_j | j$\in$ J} is fuzzy (γ, γ')-open.

(iv) If A is fuzzy γ-open, then A is fuzzy [γ, γ'] –open for any fuzzy operation γ'.

(v) If (X, T) is fuzzy γ- regular space and A is fuzzy [γ, γ']-open for a fuzzy operation γ', then A is γ-open.

***Proof*:** (i) Let p_x^λ q A $\cap$ B. Then p_x^λ q A and p_x^λ q B. Since A and B are fuzzy γ-open and fuzzy γ'-open respectively, there exist open q-neighborhoods U and V of p_x^λ such that γ(U)$\subseteq$A and γ'(V)$\subseteq$B.

Now

$$(\gamma(U) \cap \gamma'(V))(x) = \min\{ \gamma(U)(x), \gamma'(V)(x) \}$$
$$\leq \min\{ A(x), B(x) \}$$
$$= (A \cap B)(x)$$

Hence γ(U)$\cap$ γ'(V) $\subseteq$ A $\cap$ B and so A $\cap$ B is fuzzy [γ, γ']-open.

(ii) Let A be fuzzy [γ, γ']-open and p_x^λ q A. Then for p_x^λ q A, there exists Q-nbds U_x and V_x of p_x^λ such that $\gamma(U_x) \cap \gamma'(V_x) \subseteq$ A. But by definition of γ we have $U_x \subseteq \gamma(U_x)$ and $V_x \subseteq \gamma(V_x)$. Hence we have $U_x \cap V_x \subseteq$A. Since U_x and V_x open Q-neighborhoods, so $V_x \cap V_x = W_x$ (say) is also open q-neighborhood of p_x^λ. Thus we have $W_x \subseteq$A. This shows A is open.

(iii) Let B = $\bigcup$ { A_j | j∈ J } and p_x^λ q B. Then there exists some $A_j \in$ T such that p_x^λ q A_j. Since A_j is fuzzy $[\gamma,\gamma']$-open, there exists open q-neighborhoods U and V of p_x^λ such that γ(U) $\bigcap$ γ'(V) $\subseteq$ A_j. Therefore (γ(U) $\bigcap$ γ'(V))(x) $\leq$ A_j(x) and so (γ(U) $\bigcap$ γ'(V))(x) $\leq$ Sup{A_j(x) |j∈ J }. This implies γ(U) $\bigcap$ γ'(V) $\subseteq$ B and hence B is fuzzy $[\gamma,\gamma']$-open.

(iv) Let p_x^λ q A. Since A is fuzzy γ-open set there exist a open q-neighborhoods U of p_x^λ such that γ(U) $\subseteq$A

Now γ(U) $\subseteq$A

$\Rightarrow$ γ(U) (x) $\leq$ A(x)

$\Rightarrow$ min { γ(U)(x) , γ'(V)(x) }$\leq$ A(x) where V is q-neighborhood of p_x^λ and γ' is any fuzzy operation.

Hence γ(U)$\bigcap$ γ'(V) $\subseteq$ A and so A is fuzzy $[\gamma,\gamma']$-open.

(v) Let p_x^λ q A. Since A is fuzzy γ-open then for each p_x^λ q A there exist open Q-neighborhoods U and V of p_x^λ such that γ(U)$\bigcap$ γ'(V) $\subseteq$ A.

Now

γ(U)$\bigcap$ γ'(V) $\subseteq$ A

$\Rightarrow$ (γ(U)$\bigcap$ γ'(V))(x) $\subseteq$ A(x)

$\Rightarrow$ min { γ(U)(x), γ'(V)(x) }$\leq$ A(x)

$\Rightarrow$ min { U(x), V(x) }$\leq$ A(x) by definition of γ

$\Rightarrow$ U$\bigcap$V$\subseteq$A

Since U and V are open q-neighborhood of p_x^λ therefore U$\bigcap$V is also a open Q-neighborhood of p_x^λ and let U$\bigcap$V = W. Then we have W $\subseteq$A. Again since (X,T) is fuzzy γ-regular space, there exist a open q-neighborhood S of p_x^λ such that γ(S) $\subseteq$W and hence γ(S) $\subseteq$A. This shows A is fuzzy γ-open.

Definition 6.5.3: The set of all fuzzy $[\gamma,\gamma']$-open sets of (X,T) is denoted by $T_{[\gamma,\gamma']}$.

Remark 6.5.4: The following relation 6.5.5 is shown by proposition 6.5.2 (i), (ii), and (iv)

(6.5.5) : $T_\gamma \cap T_\gamma = T_\gamma \subseteq T_\gamma \cup T_{\gamma'} \subseteq T_{[\gamma,\gamma']} \subseteq T$.

Theorem 6.5.6: Let γ and γ' be fuzzy regular operations.

(i) If A and B are $[\gamma,\gamma']$-open sets, then A $\cap$ B is $[\gamma,\gamma']$-open.

(ii) $T_{[\gamma,\gamma']}$ is a fuzzy topology on X.

Proof: (i) Let p_x^λ q (A $\cap$ B). Then p_x^λ q A and p_x^λ q B. Since A and B are fuzzy $[\gamma,\gamma']$-open, there exist open q-neighborhoods U, V, W, and S of p_x^λ such that γ(U)$\cap$ γ'(V) $\subseteq$A and γ(W)$\cap$ γ'(S) $\subseteq$ B. Since γ and γ' are fuzzy regular operations, there exist open q-neighborhoods E and F of p_x^λ such that γ(E) $\subseteq$ γ(U)$\cap$ γ(W) and γ'(F) $\subseteq$ γ'(V) $\cap$ γ'(S).

Now

(γ(E)$\cap$ γ'(F))(x) = min { (γ(E)(x), γ'(F))(x) }

$\leq$ min { (γ(U)$\cap$ γ(W))(x), (γ'(V) $\cap$ γ'(S))(x) }

= min { (γ(U)$\cap$ γ'(V))(x), (γ(W)$\cap$ γ'(S))(x) }

$\leq$ min { A(x), B(x) }

Thus γ(E)$\cap$ γ'(F) $\subseteq$ A $\cap$B. This shows A $\cap$B is $[\gamma,\gamma']$-open

(ii) 0 and 1 are fuzzy $[\gamma,\gamma']$-open sets together with (i) and theorem 6.5.2 (iii) $T_{[\gamma,\gamma']}$ is fuzzy topology on X.

Definition 6.5.7: A fuzzy topological space (X, T) is said to be fuzzy $[\gamma, \gamma']$-regular space if for each fuzzy point $p_x^\lambda \in S(X)$ and every open Q-neighborhood U of p_x^λ, there exists open Q-neighborhoods W and S of p_x^λ such that γ(W) $\cap$ γ'(S) $\subseteq$ U.

Theorem 6.5.8: Let (X, T) be fuzzy topological space.

(i) (X, T) is fuzzy $[\gamma, \gamma']$-regular if and only $T_{(\gamma,\gamma')} = T$ holds.

(ii) If (X, T) is fuzzy γ-regular and fuzzy γ'-regular space, then it is fuzzy $[\gamma, \gamma']$-regular.

Proof: (i) (Necessity): Since $T_{[\gamma,\gamma']} \subseteq T$, it is sufficient to prove $T \subseteq T_{[\gamma,\gamma']}$. Let A $\in$ T and p_x^λ q A. Then A(x) > 1 - λ and so A is open q-neighborhood of p_x^λ. Since (X, T) is fuzzy $[\gamma, \gamma']$-regular , there exists open q-neighborhoods W and S of p_x^λ such that γ(W) $\cap$ γ'(S) $\subseteq$ A. Thus we have proved that for each p_x^λq A there exist open q-neighborhoods W and S of p_x^λ such that γ(W) $\cap$ γ'(S) $\subseteq$ A. This shows that A is fuzzy $[\gamma, \gamma']$-open set.

(sufficiency): Let p_x^λ be a fuzzy point in X and let V be an open Q-neighborhood of p_x^λ. Since $T_{[\gamma,\gamma']} = T$, V is fuzzy $[\gamma, \gamma']$-open set. Therefore there exists open Q-neighborhoods W and S of p_x^λ such that γ(W) $\cap$ γ'(S) $\subseteq$V. This shows (X,T) is fuzzy $[\gamma, \gamma']$-regular .

(iii) Let (X, T) be fuzzy γ-regular and fuzzy γ'-regular space. Let p_x^λ be a fuzzy point in X. Since (X, T) is fuzzy γ-regular and fuzzy γ'-regular, so for every open Q-neighborhoods U and V of p_x^λ there exist open q-neighborhoods W and S of p_x^λ such that γ(W) $\subseteq$ U and γ'(S) $\subseteq$ V.

Now

$(\gamma(W) \cap \gamma'(S))(x) = \min\{ \gamma(W)(x), \gamma'(S)(x) \}$

$\leq \min\{ U(x), V(x) \}$

$= U(x)$ or $V(x)$

Thus $\gamma(W) \cap \gamma'(S) \subseteq U$ or $\gamma(W) \cap \gamma'(S) \subseteq V$. In both cases we can say that (X, T) is fuzzy $[\gamma, \gamma']$-regular.

6.6: Fuzzy $[\gamma, \gamma']$-closures:

We introduce $[\gamma, \gamma']$-closure of a set and investigate some properties of $[\gamma, \gamma']$-closed sets.

Definition 6.6.1: A fuzzy subset A of (X, T) is said to be fuzzy $[\gamma, \gamma']$-closed set if its complement A^c is fuzzy fuzzy $[\gamma, \gamma']$-open.

Definition 6.6.2 : For a fuzzy subset A of (X, T) and $T_{[\gamma,\gamma']}$, $T_{[\gamma,\gamma']}$-Cl(A) denotes the intersection of all $[\gamma, \gamma']$-closed sets containing A i.e.

$$T_{[\gamma,\gamma']}\text{-Cl}(A) = \inf\{ F : A \subseteq F, F \in T_{[\gamma,\gamma']} \}.$$

The following theorem characterizes $T_{[\gamma,\gamma']}$-Cl(A).

Theorem 6.6.3: For a fuzzy point p_x^λ in X, $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A) if and only V q A for any $V \in T_{[\gamma,\gamma']}$ and p_x^λ q V.

Proof : We have

$p_x^\lambda \in T_{[\gamma,\gamma']}$ -Cl(A)if and only if for every fuzzy $[\gamma, \gamma']$-closed set $F \supseteq A$, $p_x^\lambda \in F$.

i.e. $p_x^\lambda \in T_{[\gamma,\gamma']}$ -Cl(A)if and only if for every fuzzy (γ, γ')-closed set $F \supseteq A$, $F(x) \geq \lambda$.

By taking complement this fact can be stated as follows:

$p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A) if and only if for every fuzzy (γ,γ')-open set V $\subseteq$ A^c, V(x) $\leq$1-λ. In other words, $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A) if and only if for every fuzzy $[\gamma,\gamma']$-open set V satisfying V(x) > 1-λ , V is not contained in A^c . i.e $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A) if and only if V q A for any V $\in T_{[\gamma,\gamma']}$ and p_x^λ q V.

Theorem 6.6.4: Let A and B be fuzzy subsets of (X,T).

(i) A $\subseteq T_{[\gamma,\gamma']}$ -Cl(A),

(ii) If A $\subseteq$ B, then $T_{[\gamma,\gamma']}$ -Cl(A) $\subseteq T_{[\gamma,\gamma']}$ -Cl(B)

(iii) A is fuzzy $[\gamma,\gamma']$-closed if and only if $T_{[\gamma,\gamma']}$ -Cl(A) = A.

(iv) $T_{[\gamma,\gamma']}$-Cl(A) is fuzzy $[\gamma,\gamma']$-closed set.

Proof: (i) It is obvious

(ii) Let $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A). Let V fuzzy (γ,γ')-open set and p_x^λ q V. Then we have V q A. Since A$\subseteq$B , then V q B. This shows $p_x^\lambda \in T_{[\gamma,\gamma']}$ -Cl(B) and hence $T_{[\gamma,\gamma']}$-Cl(A) $\subseteq T_{[\gamma,\gamma']}$-Cl(B)

(iii) (Necessity): Let A be $[\gamma,\gamma']$-closed set. Then by definition $T_{[\gamma,\gamma']}$ -Cl(A) = A.

(Sufficiency): Let $T_{[\gamma,\gamma']}$ -Cl(A) = A. We prove that A^c is fuzzy $[\gamma,\gamma']$-open set. Let p_x^λ q A^c. Then we have $p_x^\lambda \notin$ A = $T_{[\gamma,\gamma']}$-Cl(A)and consequently there exists a fuzzy $[\gamma,\gamma']$-open set V and p_x^λ q V such that V is not quasi-concident with A. Therefore we have $V \subseteq A^C$. Since V is fuzzy $[\gamma,\gamma']$-open set, so for p_x^λ q V, there exists open q-neighborhoods W and S of p_x^λ such that γ(W) $\cap$ γ'(S) $\subseteq$ V. Hence we have γ(W) $\cap$ γ'(S) $\subseteq$ A^c. This shows A^C is fuzzy $[\gamma,\gamma']$-open set and hence A is fuzzy $[\gamma,\gamma']$-closed.

(iv) Here we prove that $T_{[\gamma,\gamma']}$-Cl($T_{[\gamma,\gamma']}$-Cl(A)) = $T_{[\gamma,\gamma']}$-Cl(A). Let us put

G = $T_{[\gamma,\gamma']}$-Cl($T_{[\gamma,\gamma']}$-Cl(A)) and H = $T_{[\gamma,\gamma']}$-Cl(A). Let $p_x^\lambda \in$ G and V be fuzzy γ-open set and p_x^λ q V. Then we have V q H for each fuzzy $[\gamma,\gamma']$-open set V and p_x^λ q V. This implies V(x) +H(x) > 1 for some $x \in X$. Let H(x) = r, r∈ [0,1]. Then

$p_x^r \in$ H = $T_{[\gamma,\gamma']}$-Cl(A) and V is fuzzy $[\gamma,\gamma']$-open set and p_x^r q V. Therefore V q A and so $p_x^\lambda \in$ $T_{[\gamma,\gamma']}$-Cl(A).

Again, let $p_x^\lambda \in$ $T_{[\gamma,\gamma']}$-Cl(A).Then by (i), $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl$T_{[\gamma,\gamma']}$-Cl(A))). Thus we have shown that $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl($\tau_{[\gamma,\gamma']}$-$T_{[\gamma,\gamma']}$-Cl(A)) ⇔ $p_x^\lambda \in (T_{[\gamma,\gamma']}$-Cl(A)).

Hence $T_{[\gamma,\gamma']}$-Cl($T_{(\gamma,\gamma')}$-Cl(A)) = $T_{[\gamma,\gamma']}$-Cl(A) and by (iii) $T_{[\gamma,\gamma']}$-Cl(A) is fuzzy $[\gamma,\gamma']$-closed set.

We introduce the following definition of Cl $_{(\gamma,\gamma')}$(A).

Definition 6.6.5: A fuzzy point p_x^λ in X is in the fuzzy $[\gamma,\gamma']$-closure of fuzzy set A of X i.e. in Cl $_{(\gamma,\gamma')}$(A) if (γ(W) ⋂ γ'(S)) q A for each open q-neighborhoods V and W of p_x^λ.

Theorem 6.6.6: For a fuzzy subset A of (X,T) the following properties hold.

(i) A ⊆ Cl(A) ⊆ Cl$_{[\gamma,\gamma']}$(A) ⊆ $T_{[\gamma,\gamma']}$-Cl(A)

(ii) Cl$_{[\gamma,\gamma']}$(A) ⊆ Cl$_{(\gamma,\gamma')}$(A)

Proof: (i) Let $p_x^\lambda \in$ Cl(A). Let U and V be any open Q-neighborhood of p_x^λ. Then we have U q A and V q A. By the definition of γ, we get

γ(U) q A and γ'(V) q A. Therefore min{ γ(U)(x), γ'(V)(x) } + A(x) > 1 for some x and so (γ(W) ⋂ γ'(S)) q A. This shows that $p_x^\lambda \in$ Cl$_{[\gamma,\gamma']}$(A).

Hence Cl(A) ⊆ Cl$_{[\gamma,\gamma']}$(A). Now we show that Cl$_{[\gamma,\gamma']}$(A) ⊆ $T_{[\gamma,\gamma']}$-Cl(A).

Let $p_x^\lambda \notin T_{[\gamma,\gamma']}$-Cl(A).Then there exists an fuzzy $[\gamma,\gamma']$-open V such that p_x^λ q V and V is not quasi-concident with A. Then we have V(x) + A(x) ≤ 1. Since V is fuzzy $[\gamma,\gamma']$-open set, so there exists q-neighborhoods W and S of p_x^λ such that γ(W) ⋂ γ'(S) ⊆ V. Therefore γ(W)⋂ γ'(S) is not quasi-concident with A. Accordingly $p_x^\lambda \notin$ $Cl_{[\gamma,\gamma']}$(A). Hence $Cl_{[\gamma,\gamma']}$(A) ⊆ $T_{[\gamma,\gamma']}$-Cl(A).

Thus we have A⊆ Cl(A) ⊆ $Cl_{[\gamma,\gamma']}$(A) ⊆ $T_{[\gamma,\gamma']}$-Cl(A).

(ii) Let $p_x^\lambda \in$ $Cl_{[\gamma,\gamma']}$(A). Then we have (γ(W) ⋂ γ'(S)) q A for every open Q-neighborhoods W and S of p_x^λ.

Now

(γ(W) ⋂ γ'(S)) q A

⇒ min { γ(W)(x), γ'(S)(x) } + A(x) > 1 for some x

⇒ max { γ(W)(x), γ'(S)(x) } + A(x) > 1

⇒ (γ(W) ⋃ γ'(S)) q A

Thus $p_x^\lambda \in$ $Cl_{(\gamma,\gamma')}$ (A) and $Cl_{[\gamma,\gamma']}$(A) ⊆ $T_{[\gamma,\gamma']}$-Cl(A).

Theorem 6.6.7: Let A be a fuzzy subset of (X,T).

(i) A is fuzzy $[\gamma,\gamma']$-closed if and only if $Cl_{[\gamma,\gamma']}$(A). = A.

(ii) $T_{[\gamma,\gamma']}$-Cl(A) = A if and only if $Cl_{[\gamma,\gamma']}$(A). =A.

(iii) A is fuzzy $[\gamma,\gamma']$-open if and only if $Cl_{[\gamma,\gamma']}(A^c)$. = A^c

Proof: (i) (Necessity): we prove that $Cl_{[\gamma,\gamma']}$(A). ⊆ A. Let $p_x^\lambda \notin A$. Then p_x^λ q A^C. Since A^C is fuzzy $[\gamma,\gamma']$-open, there exist open Q-neighborhoods W and S of p_x^λ such that γ(W) ⋂ γ(S) ⊆ A^c and so γ(W)⋂ γ(S) is not quasi-concident with A. Therefore $p_x^\lambda \notin$ $Cl_{[\gamma,\gamma']}$(A) and $Cl_{[\gamma,\gamma']}$(A). ⊆ A. Again by theorem 6.6.6(i) we have

$A \subseteq Cl_{[\gamma,\gamma']}(A)$.Thus we get $Cl_{[\gamma,\gamma']}(A)$.= A.

(Sufficiency): We prove that A^c is fuzzy $[\gamma,\gamma']$-open. Let p_x^λ q A^c .Then

$p_x^\lambda \notin A = Cl_{[\gamma,\gamma']}(A)$ and consequently there exists fuzzy open Q-neighborhoods W

and S of p_x^λ such that γ(W) ⋂ γ(S) is not quasi-concident with A. Hence we have

γ(W) ⋂ γ(S) $\subseteq A^c$.This shows that A^c is fuzzy $[\gamma,\gamma']$-open i.e. A is $[\gamma,\gamma']$-closed.

(ii). It is proved by (i) and theorem 6.6.4 (iii).

(iii) It follows from (i) and definition 6.6.1

Theorem 6.6.8: For a fuzzy subset A of (X,τ) the following properties hold.

(i) If (X,T) is fuzzy $[\gamma,\gamma']$-regular space then Cl(A) = $Cl_{[\gamma,\gamma']}(A)$= $T_{[\gamma,\gamma']}$-Cl(A)

(ii) $Cl_{[\gamma,\gamma']}(A)$ is fuzzy closed subset of (X,T).

(iii) $T_{[\gamma,\gamma']}$-Cl($Cl_{[\gamma,\gamma']}(A)$) = $T_{[\gamma,\gamma']}$-Cl(A) = $Cl_{[\gamma,\gamma']}$ ($T_{[\gamma,\gamma']}$-Cl(A))

(iv) If $A \subseteq B$ then $Cl_{[\gamma,\gamma']}(A) \subseteq Cl_{[\gamma,\gamma']}(B)$

Proof: (i) By theorem 6.5.8(i), we have T =$T_{[\gamma,\gamma']}$ and hence Cl(A)) = $T_{[\gamma,\gamma']}$-Cl(A). By using theorem 6.6.6(i) it is shown that Cl(A) = $Cl_{[\gamma,\gamma']}(A)$= $T_{[\gamma,\gamma']}$-Cl(A).

(ii) Here we want to prove that Cl($Cl_{[\gamma,\gamma']}(A)$) = $Cl_{[\gamma,\gamma']}(A)$. Since

$Cl_{[\gamma,\gamma']}(A) \subseteq$ Cl($Cl_{[\gamma,\gamma']}(A)$, it is require to prove that Cl ($Cl_{[\gamma,\gamma']}(A)$) $\subseteq Cl_{[\gamma,\gamma']}(A)$.

Let $p_x^\lambda \in$ Cl($Cl_{[\gamma,\gamma']}(A)$). Let U and V be any open q-neighborhood of p_x^λ.Then we have U q $Cl_{[\gamma,\gamma']}(A)$ and V q $Cl_{[\gamma,\gamma']}(A)$.

Therefore

$\min\{ (U)(x), (V)(x) \} + Cl_{[\gamma,\gamma']}(A)(x) > 1$

$\Rightarrow \min\{ (U)(x), (V)(x) \} + r > 1$ where $Cl_{[\gamma,\gamma']}(A)(x) = r$, $r \in [0,1]$.

$\Rightarrow U(x) + r > 1$ and $V(x) + r > 1$

$\Rightarrow$ U(x) > 1 – r and V(x) > 1 – r

$\Rightarrow$ U and V are open q-neighborhood of p_x^r.

Also $p_x^r \in \mathrm{Cl}_{[\gamma,\gamma']}$(A). Therefore by definition 6.6.5 we have

(γ(U) $\cap$ γ'(V)) q A and so $p_x^\lambda \in \mathrm{Cl}_{[\gamma,\gamma']}$(A). Thus

Cl (Cl$_{[\gamma,\gamma']}$(A)) $\subseteq$ Cl$_{[\gamma,\gamma']}$(A).

(iii) By the theorem 6.6.4 (iv), we have $T_{[\gamma,\gamma']}$-Cl(A) is fuzzy $[\gamma,\gamma']$-closed subset of X.

Then by theorem 6.6.7 (i) we get $T_{[\gamma,\gamma']}$-Cl(A)= Cl$_{[\gamma,\gamma']}$ ($T_{[\gamma,\gamma']}$-Cl(A)).

Since A$\subseteq$ Cl$_{[\gamma,\gamma']}$(A), $T_{[\gamma,\gamma']}$-Cl(A) $\subseteq$ $T_{[\gamma,\gamma']}$-Cl(Cl$_{[\gamma,\gamma']}$(A)).

Then by theorem 6.6.6(i) we obtain that

Cl$_{[\gamma,\gamma']}$(A) $\subseteq T_{[\gamma,\gamma']}$-Cl(A) $\subseteq$ $T_{[\gamma,\gamma']}$-Cl(Cl$_{[\gamma,\gamma']}$(A)).

By using these inclusions and theorem 6.6.4(ii) we have

$T_{[\gamma,\gamma']}$-Cl(Cl$_{[\gamma,\gamma']}$(A)) $\subseteq T_{[\gamma,\gamma']}$-Cl($T_{[\gamma,\gamma']}$-Cl(A)) $\subseteq T_{[\gamma,\gamma']}$-Cl($T_{[\gamma,\gamma']}$-Cl(Cl$_{[\gamma,\gamma']}$(A)).

By theorem 6.6.4(iv) and 6.6.4(iii) it can be written as

$T_{[\gamma,\gamma']}$-Cl (Cl$_{[\gamma,\gamma']}$(A)) $\subseteq$ $T_{[\gamma,\gamma']}$-Cl(A) $\subseteq$ $T_{[\gamma,\gamma']}$-Cl(Cl$_{[\gamma,\gamma']}$(A)).

Thus $T_{[\gamma,\gamma']}$-Cl(A) = $T_{[\gamma,\gamma']}$-Cl (Cl$_{[\gamma,\gamma']}$(A))

and hence $T_{[\gamma,\gamma']}$-Cl (Cl$_{[\gamma,\gamma']}$(A)) = $T_{[\gamma,\gamma']}$-Cl(A) = Cl$_{[\gamma,\gamma']}$ ($T_{[\gamma,\gamma']}$-Cl(A)).

(iv) It is obious.

Theorem 6.6.9: Let γ and γ' be open operations and A a fuzzy subset of (X,T).

Then the followings hold:

(i) Cl$_{[\gamma,\gamma']}$(A) = $T_{[\gamma,\gamma']}$-Cl(A) and

(ii) Cl$_{[\gamma,\gamma']}$(Cl$_{[\gamma,\gamma']}$(A)) = Cl$_{[\gamma,\gamma']}$(A)i.e. Cl$_{[\gamma,\gamma']}$(A) is fuzzy (γ,γ')-closed set..

Proof: (i) By theorem 6.6.6(i), it suffices to prove that $T_{[\gamma,\gamma']}$-Cl(A) $\subseteq$ Cl$_{[\gamma,\gamma']}$(A).

Let $p_x^\lambda \in T_{[\gamma,\gamma']}$-Cl(A). Let U and V be open q-neighborhood of p_x^λ.By the openness of γ and γ', there exists a fuzzy γ-open set W and γ'-open set S such that p_x^λ q W and W $\subseteq \gamma$(U) and p_x^λ q S and S $\subseteq \gamma'$(V). By theorem 6.5.2 (i), W $\cap$ S is fuzzy $[\gamma,\gamma']$-open set and then by theorem 6.6.3, we have (W $\cap$ S) q A and hence (γ(U) $\cap$ γ'(V)) q A. This shows $p_x^\lambda \in$ Cl$_{[\gamma,\gamma']}$(A). Thus $T_{[\gamma,\gamma']}$-Cl(A) $\subseteq$ Cl$_{[\gamma,\gamma']}$(A).

(ii) This follows immediately from (i) and theorem 6.6.8(iii).

6.7. Fuzzy $[\gamma,\gamma']$-separations axioms:

In this section we introduce fuzzy $[\gamma,\gamma']$-g-closed set and fuzzy $[\gamma,\gamma']-T_i(i=1,2,\frac{1}{2})$ spaces and obtain their properties.

Throughout this section, let γ and γ' be given two fuzzy operations on fuzzy topology T and $X\times X$ the fuzzy product of X and $\Delta(X)=\{(p_x^\lambda,p_x^\lambda):p_x^\lambda\in S(X)\}$.

Definition 6.7.1: A fts (X,T) is called fuzzy $[\gamma,\gamma']$-T_1 iff for each $(p_x^\lambda,p_y^k)\in X\times X-\Delta(X)$,there exists open Q-neighbourhoods U , V of p_x^λ and W,S of p_y^k such that $p_y^k\overline{q}(\gamma(U)\cap\gamma'(V))$ and $p_x^\lambda\overline{q}(\gamma(W)\cap\gamma'(S))$

Definition 6.7.2: A fts (X,T) is called fuzzy (γ,γ')-T_2 iff for each $(p_x^\lambda,p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U , V of p_x^λ and W, S of p_y^k such that $(\gamma(U)\cap\gamma'(V))\overline{q}(\gamma(W)\cap\gamma'(S))$.

Theorem 6.7.3: A space (X,T) is fuzzy $[\gamma,\gamma']$-T_1 if and only any fuzzy singleton in X is a fuzzy $[\gamma,\gamma']$-closed set.

Proof: (Necessity): Let (X,T) be a fuzzy $[\gamma,\gamma']$-T_1 and $p_x^\lambda\in$ S(X). Since $p_x^\lambda \subseteq cl_{[\gamma,\gamma']}(p_x^\lambda)$, so it is only need to prove $cl_{[\gamma,\gamma']}(p_x^\lambda)\subseteq p_x^\lambda$. Let $p_y^k\notin p_x^\lambda$.Then for each $(p_x^\lambda,p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U , V of p_x^λ and W, S of p_y^k such that $p_y^k\overline{q}(\gamma(U)\cap\gamma'(V))$ and $p_x^\lambda\overline{q}(\gamma(W)\cap\gamma'(S))$. Since $p_y^k\overline{q}(\gamma(U)\cap\gamma'(V))$ means $p_y^k\notin cl_{[\gamma,\gamma']}(p_x^\lambda)$, thus $cl_{[\gamma,\gamma']}(p_x^\lambda)\subseteq p_x^\lambda$.

(sufficiency): Let $p_x^\lambda, p_y^k\in$ S(X) and $p_x^\lambda\neq p_y^k$. Since p_x^λ and p_y^k are both $[\gamma,\gamma']$-closed set, $cl_{[\gamma,\gamma']}(p_x^\lambda)=p_x^\lambda$ and $cl_{[\gamma,\gamma']}(p_y^r)=p_y^r$. Since $p_x^\lambda\neq p_y^k$, then $p_y^k\notin cl_{[\gamma,\gamma']}(p_x^\lambda)$ and $p_x^\lambda\notin cl_{[\gamma,\gamma']}(p_x^\lambda)$. Therefore, there exists open Q-neighbourhoods U,V of p_x^λ and W, S Open Q-nbds of p_y^k such that $(\gamma(U)\cap\gamma'(V))\overline{q}p_y^k$ and $(\gamma(W)\cap\gamma'(S))\overline{q}p_x^\lambda$.Thus for each $(p_x^\lambda, p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U, V of p_x^λ and W,S of p_y^k such that $(\gamma(U)\cap\gamma'(V))\overline{q}p_y^k$ and $(\gamma(W)\cap\gamma'(S))\overline{q}p_x^\lambda$. This implies (X,T) is a fuzzy $[\gamma,\gamma']$-T_1 space.

Theorem 6.7.4: If (X,T) is fuzzy $[\gamma,\gamma']$-T_2, then it is fuzzy $[\gamma,\gamma']$-T_1.

Proof: Let (X,T) be a fuzzy $[\gamma,\gamma']$-T_2 space. Then for each $(p_x^\lambda, p_y^k)\in X\times X-\Delta(X)$, there exists open Q-neighbourhoods U , V of p_x^λ and W,S of p_y^k such that $(\gamma(U)\cap\gamma'(V))\overline{q}(\gamma(W)\cap\gamma'(S))$.

Since $p_x^\lambda\neq p_y^k$, $p_x^\lambda q(\gamma(U)\cap\gamma'(V))$ and $p_y^k q(\gamma(W)\cap\gamma'(S))$, then $(\gamma(U)\cap\gamma'(V))\overline{q}p_y^k$ and $(\gamma(W)\cap\gamma'(S))\overline{q}p_x^\lambda$. Hence (X,T) is fuzzy $[\gamma,\gamma']$-T_1

Definition 6.7.5: Let (X,T) be a fts and γ an operation on T. A fuzzy set $A\in I^X$ is called $[\gamma,\gamma']$-generalized closed ($[\gamma,\gamma']$-g.closed, for short) if $cl_{[\gamma,\gamma']}(A)\subseteq U$ whenever $A\subseteq U$ and U is fuzzy $[\gamma,\gamma']$-open in (X,T).

Theorem 6.7.6: Every fuzzy $[\gamma,\gamma']$-closed set is fuzzy $[\gamma,\gamma']$-g.closed.

Proof: It is obvious.

Definition 6.7.7: A space (X,T) is called a fuzzy $[\gamma,\gamma']$-$T_{1/2}$ space if every fuzzy $[\gamma,\gamma']$-g.closed set of (X,T) is fuzzy $[\gamma,\gamma']$-closed

Theorem 6.7.8: For each $p_x^\lambda \in$ S(X), p_x^λ is $[\gamma,\gamma']$-closed or $(p_x^\lambda)^C$ is fuzzy $[\gamma,\gamma']$-g.closed set in (X,T).

Proof: Suppose p_x^λ is not $[\gamma,\gamma']$-closed. Then $(p_x^\lambda)^C$ is fuzzy $[\gamma,\gamma']$-open. Let U be any fuzzy $[\gamma,\gamma']$-open set such that $(p_x^\lambda)^C \subseteq U$. Since $U = X$ is the only fuzzy $[\gamma,\gamma']$-open, $cl_{[\gamma,\gamma']}((p_x^\lambda)^C) \subseteq U$. Therefore $(p_x^\lambda)^C$ is fuzzy $[\gamma,\gamma']$-g-closed set.

6.8. Fuzzy ([γ,γ'], [β,β'])-continuous mapping:

Throughout this section, let f:(X,T)→(Y,T') be fuzzy mapping and let γ,γ': T → I^X be fuzzy operation on T and β,β': T' → I^Y be fuzzy operation on T'.

Definition 6.8.1: A mapping f:(X,T)→(Y,T') is said to be fuzzy ([γ,γ'],[β,β'])-continuous if and only for every fuzzy point p_x^λ in X and every fuzzy open Q-neighborhood W and S of f(p_x^λ), there exists a fuzzy open Q-neighborhood U and V of p_x^λ such that f(γ(U) ∩ γ'(V) ⊆ β(W)∩ β'(S)

Theorem 6.8.2: Let (i), (ii), (iii) and (iv) be the following properties for a fuzzy mapping f:(X,T)→ (Y,T').

(i) f:(X,T)→ (Y,T') is fuzzy ([γ,γ'],[β,β'])-continuous mapping.

(ii) f($Cl_{[\gamma,\gamma']}$(A)) ⊆ $Cl_{[\beta,\beta']}$(f(A)) for every fuzzy subset A of (X,τ).

(iii) For any fuzzy [β,β']-closed set B of (Y,T'), f^{-1}(B) is fuzzy [γ,γ']-closed set in (X,T).

(iv) For any B∈ $T'_{[\beta,\beta']}$, f^{-1}(B) ∈ $T_{[\gamma,\gamma']}$ holds.

Then (i) ⇒ (ii) ⇒ (iii) ⇒(iv)

Proof: (i) $\Rightarrow$ (ii). Let $p_x^\lambda \in cl_{[\gamma,\gamma']}(A)$ and let W and S be open Q-neighbourhood of $f(p_x^\lambda)$. Then there exists open Q-neighborhoods U and V of p_x^λ such that

$f(\gamma(U) \cap \gamma'(V) \subseteq \beta(W) \cap \beta'(S)$ .Again $p_x^\lambda \in Cl_{[\gamma,\gamma']}(A)$, $\Rightarrow (\gamma(U) \cap \gamma'(V))$ q A $\Rightarrow$

$f(\gamma(U) \cap \gamma'(V))$ q $f(A)$ $\Rightarrow$ $(\beta(W) \cap \beta'(S))$ q $f(A)$ $\Rightarrow$ $f(p_x^\lambda) \in Cl_{[\beta,\beta']}(f(A))$ $\Rightarrow$

$p_x^\lambda \in f^{-1}(cl_{[\beta,\beta']}(f(A)))$. Thus $cl_{[\gamma,\gamma']}(A) \subseteq f^{-1}(cl_{[\beta,\beta']}(f(A)))$ so that $f(cl_{[\gamma,\gamma']}(A)) \subseteq cl_{[\beta,\beta']}(f(A))$

(ii) $\Rightarrow$ (iii). Let B be a fuzzy $[\beta,\beta']$-closed set of (Y,T'). Then

$Cl_{[\beta,\beta']}(B) = B$. By using (i) we have

$f(Cl_{[\gamma,\gamma']}(f^{-1}(B))) \subseteq Cl_{[\beta,\beta']}(ff^{-1}(B)) \subseteq Cl_{[\beta,\beta']}(B) = B$. Thus $Cl_{[\gamma,\gamma']}(f^{-1}(B)) \subseteq f^{-1}(B)$.

Again by proposition 6.6.6(i) we have $f^{-1}(B) \subseteq Cl_{[\gamma,\gamma']}(f^{-1}(B))$. Hence

$Cl_{[\gamma,\gamma']}(f^{-1}(B)) = f^{-1}(B)$. That is $f^{-1}(B)$ is fuzzy $[\gamma,\gamma']$-closed set in (X,T).

(iii) $\Rightarrow$ (iv). Let B be fuzzy β-open set in Y. Then B^c is fuzzy $[\beta, \beta']$-closed set in Y. Then by (ii) $f^{-1}(B^c) = (f^{-1}(B))^c$ is fuzzy $[\gamma,\gamma']$-closed set in X and hence $f^{-1}(B)$ is fuzzy $[\gamma,\gamma']$-open set in X.

Theorem 6.8.3. Let f:(X,T)$\rightarrow$(Y,T') be fuzzy mapping and (Y,T') fuzzy -regular space, then following statements are equivalent.

(i) f:(X,T)$\rightarrow$(Y,T') is fuzzy ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous mapping.

(ii) $f(Cl_{[\gamma,\gamma']}(A)) \subseteq Cl_{[\beta,\beta']}(f(A))$ holds for every fuzzy subset A of (X,T).

(iii) For any fuzzy $[\beta, \beta']$-closed set of (Y,T'), $f^{-1}(B)$ is fuzzy $[\gamma,\gamma']$- closed in X.

Proof: By theorem 6.8.2, we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii), so it is sufficient to prove (iii)$\Rightarrow$(i). Let p_x^λ be a fuzzy point in X and W and S be a fuzzy open q-neighborhood of $f(p_x^\lambda)$

Since W$\cap$S is also a open Q-neighborhood of $f(p_x^\lambda)$, then by theorem 6.5.8, W$\cap$S is fuzzy $[\beta,\beta']$-open set in Y and hence $(W\cap S)^C$ is fuzzy $[\beta,\beta']$-closed set in Y. Then by our assumption, $f^{-1}((W\cap S)^C) = (f^{-1}(W\cap S))^C$ is fuzzy $[\gamma,\gamma']$-closed set in X. Therefore $f^{-1}(W\cap S)$ is fuzzy $[\gamma,\gamma']$-open set in X. Also we have $f(p_x^\lambda)$ q (W$\cap$S). This implies p_x^λ q f^{-1}(W$\cap$S). Since f^{-1}(W$\cap$S) is fuzzy $[\gamma,\gamma']$-open set , there exists open Q-neighborhoods U and V of p_x^λ such that γ(U) $\cap$ γ'(V) $\subseteq$ f^{-1}(W$\cap$S) and hence

$f(\gamma(U) \cap \gamma'(V)) \subseteq W\cap S \subseteq \beta(W) \cap \beta'(S)$ so that f is fuzzy

$([\gamma,\gamma'],[\beta,\beta'])$-continuous.

Theorem 6.8.4: Let f:(X,T)$\rightarrow$(Y,T') be fuzzy mapping and β and β' be fuzzy open operation, then following statements are equivalent.

(i) f:(X,T)$\rightarrow$(Y,T') is fuzzy $([\gamma,\gamma'],[\beta,\beta'])$ continuous.

(ii) $f(\mathrm{Cl}_{[\gamma,\gamma']}(A)) \subseteq \mathrm{Cl}_{[\beta,\beta']}(f(A))$ holds for every fuzzy subset A of (X,T).

(iii) For any fuzzy $[\beta,\beta']$-closed set of (Y,T), f^{-1}(B) is fuzzy $[\gamma,\gamma']$- closed in X.

Proof: By theorem 6.8.2 we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), so it is sufficient to prove that (iii)$\Rightarrow$(i). Let p_x^λ be a fuzzy point in X and V be a fuzzy open q-neighborhood of $f(p_x^\lambda)$. Since β and β' are fuzzy open operations, there exists a fuzzy β-open set A and a fuzzy β'-open set B and $f(p_x^\lambda)$ q A and $f(p_x^\lambda)$ q B such that A $\subseteq$ β(W) and B$\subseteq$ β'(S). Hence $f(p_x^\lambda)$ q (A $\cap$B) and A $\cap$B $\subseteq \beta$(W)$\cap$ β'(S) .Again since A $\cap$B is fuzzy $[\beta,\beta']$-open set in Y , (A $\cap$B)^c is fuzzy $[\beta,\beta']$ -closed set in Y. Then by our assumption, $f^{-1}((A\cap B)^c) = (f^{-1}(A\cap B))^c$ is fuzzy $[\gamma,\gamma']$-closed set in X. Consequently f^{-1}(A $\cap$B) is fuzzy $[\gamma,\gamma']$-open set in X and p_x^λ q f^{-1}(A $\cap$B).Then there exist open Q-neighborhoods U and V of p_x^λ such that γ(U)$\cap$ γ'(V) $\subseteq$ f^{-1}(A $\cap$B) and hence

$f(\gamma(U)\cap \gamma'(V)) \subseteq A\cap B \subseteq \beta(W)\cap \beta'(S)$ This shows that f is fuzzy

$([\gamma,\gamma'],[\beta,\beta'])$-continuous mapping.