CHAPTER-5
Fuzzy (γ,β)-continuous mapping and Fuzzy (γ,β)-closed(open) mapping.

5.1. Introduction:

In this chapter, we have defined a new class of continuous functions called fuzzy (γ,β)-continuous functions that generalizes several forms fuzzy continuity viz.fuzzy continuity, fuzzy θ-continuity, fuzzy δ-continuity,fuzzy weak-continuity, fuzzy strong θ-continuity,fuzzy super continuity and fuzzy weak θ-continuity. Then we have introduced the notion of fuzzy (γ,β)-open, and fuzzy (γ,β)-closed mappings which generalizes the concepts of fuzzy open(closed), fuzzy θ-open(fuzzy θ-closed) and fuzzy δ-open(fuzzy δ-closed) mappings. After that we have introduced the concepts of fuzzy (γ,β)-homeomorphism and particularly, fuzzy homeomorphism, fuzzy θ-homeomorphism and fuzzy δ-homeomorphism. Several characterizations of these mappings are also investigated.

Throughout this chapter, let $f:(X,T)\to(Y,T')$ be fuzzy mapping and let $\gamma:T\to I^X$ be operation on T and $\beta:T'\to I^Y$ be operation on T'.

5.2. Fuzzy (γ, β)-continuous mapping:

In this section we begin with the concepts of fuzzy (γ,β)-continuous mapping and discuss some some of their properties.

Defination 5.2.1: A mapping f:(X,T)→(Y,T') is said to be fuzzy (γ,β)-continuous if and only if for every fuzzy point p_x^λ in X and every fuzzy open Q-neighborhood V of $f(p_x^\lambda)$, there exists a fuzzy open Q-neighborhood U of p_x^λ such that $f(\gamma(U)) \subseteq \beta(V)$.

Examples 5.2.2:

(1) For $\gamma = \beta =$ identity operation, fuzzy (γ, β)-continuity coincides with fuzzy continuity [21, 83]

(2) For $\gamma = \beta =$ closure operation, fuzzy (γ, β)-continuity coincides with fuzzy θ-continuity [64]

(3) For $\gamma =$ identity operation and $\beta =$ closure operation, then (γ, β)-continuity coincides with fuzzy weakly θ-continuity [64]

(4) For $\gamma =$ closure operation and $\beta =$ identity operation, then (γ, β)-continuity coincides with fuzzy strongly θ-continuity [66]

(7) For $\gamma =$ identity operation and $\beta =$ interior-closure operation, then (γ, β)-continuity coincides with fuzzy almost continuity [38, 65]

(8) For $\gamma =$ closure operation and $\beta =$ interior-closure operation, then (γ, β)-continuity coincides with fuzzy almost strong θ-continuity [64]

(9) For $\gamma =$ interior-closure operation and $\beta =$ identity operation, then (γ, β)-continuity coincides with fuzzy super-continuity [66]

(10) For $\gamma =$ interior-closure operation and $\beta =$ closure operation, then (γ, β)-continuity coincides with fuzzy weak δ-continuity [64]

Theorem 5.2.3: Let (i), (ii), (iii) and (iv) be the following properties for a fuzzy mapping $f : (X, T) \to (Y, T')$

(i) $f : (X, T) \to (Y, T')$ is fuzzy (γ,β)-continuous mapping.

(ii) $f(\mathrm{Cl}_\gamma(A)) \subseteq \mathrm{Cl}_\beta(f(A))$ for every fuzzy subset A of (X,T).

(iii) For any fuzzy β-closed set B of (Y, T'), $f^{-1}(B)$ is fuzzy γ-closed set in (X,T).

(iv) For any fuzzy β-open set B of (Y, T'), $f^{-1}(B)$ is fuzzy γ-open set in (X,T).

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv)

Proof: (i) $\Rightarrow$ (ii). Let $p_x^{\lambda} \in cl_{\gamma}(A)$ and let V be an open Q-nbd. of $f(p_x^{\lambda})$. Since f is fuzzy (γ,β)-continuous, there exists an open Q-nbd. of p_x^{λ} such that $f(\gamma(U)) \subseteq \beta(V)$.

Now $p_x^{\lambda} \in cl_{\gamma}(A) \Rightarrow \gamma(U)qA \Rightarrow f(\gamma(U))qf(A) \Rightarrow \beta(V)qf(A)$

$\Rightarrow f(p_x^{\lambda}) \in cl_{\beta}(f(A)) \Rightarrow p_x^{\lambda} \in f^{-1}(cl_{\beta}(f(A)))$.

Thus $cl_{\gamma}(A) \subseteq f^{-1}(cl_{\beta}(f(A)))$ so that $f(cl_{\gamma}(A)) \subseteq cl_{\beta}(f(A))$

(ii) $\Rightarrow$ (iii) Let B be a fuzzy β-closed set of (Y,T'). Then $cl_{\beta}(B) = B$

and hence by (i), $f(cl_{\gamma}(f^{-1}(B))) \subseteq cl_{\beta}(ff^{-1}(B)) \subseteq cl_{\beta}(B) = B$,

whence we do obtain $cl_{\gamma}(f^{-1}(B)) \subseteq f^{-1}(B)$.

Thus $cl_{\gamma}(f^{-1}(B)) = f^{-1}(B)$ and hence $f^{-1}(B)$ is fuzzy γ-closed set in X.

(iii) $\Rightarrow$ (iv). Let B be fuzzy β-open set in Y. Then B^c is fuzzy β-closed set in Y. Then by (iii), $f^{-1}(B^c)$ is fuzzy γ-closed set in X. Since $f^{-1}(B^c) = 1 - f^{-1}(B)$, f^{-1}(B) is fuzzy γ-open set in X.

Corollary 5.2.4: If (Y,T') fuzzy β-regular space, then all the properties of the theorem 5.2.3 are equivalent.

Proof: By Theorem 5.2.3 we have (i) $\Rightarrow$(ii) $\Rightarrow$(iii)$\Rightarrow$(iv), so it is sufficient to prove (iv)$\Rightarrow$(i). Let p_x^{λ} be a fuzzy point in X and V be a fuzzy open Q-neighborhood of $f(p_x^{\lambda})$ .Since (Y,T') is fuzzy β-regular space, then by proposition 3.2.15, V is fuzzy β-open set in Y. By hypothesis, $f^{-1}(V)$ is fuzzy γ-open set in X. Also we have p_x^{λ} q f^{-1}(V). Since $f^{-1}(V)$ is fuzzy γ-open set, there exists an open Q-neighbourhood U of p_x^{λ} such that γ(U) $\subseteq f^{-1}$(V) and so $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Thus f is fuzzy (γ,β)-continuous.

Remark 5.2.5: The β-regularity on the codomain space of above Corollary 5.2.4 can not be removed as shown by the following example.

Example 5.2.6: Let $X = \{x, y\}$ and A, B , C $\in I^X$ defined by

A = $\underline{0.6}$, B (x) = .6, B (y) = 0.7, C = $\underline{0.3}$,

where $\underline{\alpha}$ denotes the constant mapping with value α.

Let $T = \{X ,\varnothing A, B, C\}$ and $T' = \{X ,\varnothing, B, C\}$

Then (X,T) and (X,T') are fts and (X,T') is not β-regular space.

Define $\gamma : T \to I^X$ by $\gamma(X) = X, \gamma(\varnothing) = \varnothing,\ \gamma(A) = A, \gamma(B) = B, \gamma(C) = \underline{0.5}$.

and $\beta : T' \to I^X$ by $\beta(X) = X, \beta(\varnothing) = \varnothing, \beta(B) = B, \beta(C) = \underline{0.4}$.

Now consider the identity mapping $f : (X,T) \to (X,T')$. Then the inverse image of each β-open in X (codomain) is γ-open in X (domain) but f is not fuzzy (γ, β)-continuous. For $\lambda = 0.8$ and an open Q-neighbourhood C of $f(p_x^\lambda)$, there exists no open Q-neighbourhood U of p_x^λ such that $f(\gamma(U)) \subseteq \beta(C)$.

Theorem 5.2.7: For the mapping $f : (X,T) \to (Y,T')$ the following are equivalent

(1) $f : (X,T) \to (Y,T')$ is fuzzy (γ, β)-continuous mapping.

(2) $f^{-1}(U) \subseteq \text{int}_\gamma(f^{-1}(\beta(U)))\ \ \forall\ U \in T'$

(3) $f(\text{Cl}_\gamma(A)) \subset \text{Cl}_\beta(f(A))\ \ \forall\ A \in I^X$

(4) $cl_\gamma(f^{-1}(A)) \subseteq f^{-1}(cl_\beta(A))\ \ \forall\ A \in I^Y$

(4) $f^{-1}(\text{int}_\beta(A)) \subseteq \text{int}_\gamma(f^{-1}(A))\ \forall\ A \in I^Y$

Proof: $(1) \Rightarrow (2)$: Let $U \in T'$ and $p_x^\lambda q f^{-1}(U)$. So, $f(p_x^\lambda) q U$. Since f is fuzzy (γ,β)-continuous, there exists an open Q-neighborhood V of p_x^λ such that $f(\gamma(V)) \subseteq \beta(U)$ and hence $\gamma(V)) \subseteq f^{-1}(\beta(U))$. By the definition 3.2.5, it implies that $p_x^\lambda q\, \text{int}_\gamma(f^{-1}(\beta(U)))$. Thus $p_x^\lambda q f^{-1}(U) \Rightarrow p_x^\lambda q\, \text{int}_\gamma(f^{-1}(\beta(U)))$.It follows that $f^{-1}(U) \subseteq \text{int}_\gamma(f^{-1}(\beta(U)))$.

$(2)\Rightarrow(3)$: Let $A\in I^X$ and $f(p_x^\lambda)\notin cl_\beta(f(A))$. Then there exists an open Q-neighbourhood V of $f(p_x^\lambda)$ such that $\beta(V)\vec{q}f(A)$ and hence $f^{-1}(\beta(V))\vec{q}A$. Also $f(p_x^\lambda)qV$ implies $p_x^\lambda qf^{-1}(V)$. Then by (2) we obtain that $p_x^\lambda q\,\text{int}_\gamma(f^{-1}(\beta(V)))$. Hence by definition 3.2.5, there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U)\subseteq f^{-1}(\beta(V))$. Then $\gamma(U)\overline{q}A$ and so $p_x^\lambda\notin cl_\gamma(A)$. This implies

$f(p_x^\lambda)\notin f(cl_\gamma(A))$. Thus $f(\text{Cl}_\gamma(\text{A}))\subset\text{Cl}_\beta(f(\text{A}))$.

$(3)\Rightarrow(4)$: Let $A\in I^Y$. Since $ff^{-1}(A)\subseteq A$, by theorem 3.3.7(v),

we have $cl_\beta(ff^{-1}(A))\subseteq cl_\beta(A)$. Also $f^{-1}(A)\in I^X$.

Then by (3), we have $f(cl_\gamma(f^{-1}(A))\subseteq cl_\beta(ff^{-1}(A))\subseteq cl_\beta(A)$.

Thus $cl_\gamma(f^{-1}(A))\subseteq f^{-1}(cl_\beta(A))$.

$(4)\Rightarrow(5)$: Let $A\in I^Y$ and $p_x^\lambda qf^{-1}(\text{int}_\beta(A))$.

Then $p_x^\lambda\notin(f^{-1}(in_\beta(A)))^C=f^{-1}(cl_\beta(A^C))$.

By (4), $p_x^\lambda\notin cl_\gamma(f^{-1}(A^C))=(\text{int}_\gamma(f^{-1}(A)))^C$

and hence $p_x^\lambda\overline{q}\,\text{int}_\gamma(f^{-1}(A))$. Thus $f^{-1}(\text{int}_\beta(A))\subseteq\text{int}_\gamma(f^{-1}(A))$.

$(5)\Rightarrow(1)$ Let $p_x^\lambda\in S(X)$ and V be an open Q-neighbourhood of $f(p_x^\lambda)$. Since $\beta(V)\overline{q}(\beta(V))^C$, we have $f(p_x^\lambda)\notin cl_\beta(\beta(V))^C=(\text{int}_\beta(\beta(V)))^C$ and hence $f(p_x^\lambda)q\,\text{int}_\beta(\beta(V))$ which implies $p_x^\lambda qf^{-1}(\text{int}_\beta(\beta(V)))$. By (5) , we obtain that $p_x^\lambda q\,\text{int}_\gamma(f^{-1}(\beta(V)))$. This means that there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U)\subseteq f^{-1}(\beta(V))$ and so $f(\gamma(U))\subseteq\beta(V)$. This shows that f is fuzzy (γ,β)-continuous mapping.

5.3. Fuzzy (γ,β)-open mapping and (γ,β)-closed mapping.

This section is devoted to introduction and study of the concepts of fuzzy (γ,β)-open (fuzzy (γ,β)-closed) mapping and some of their properties in fuzzy topological space

Definition 5.3.1: Let γ: T → I^X be fuzzy operation on T and β: T'→ I^Y be fuzzy operation on T'. A mapping $f:(X,T)\to(Y,T')$ is called

(1) Fuzzy (γ,β)-open if for any γ-open set A of (X,T), $f(A)$ is a β-open set.

(2) Fuzzy (γ,β)-closed if for any γ-closed set A of (X,T), $f(A)$ is a β-closed set.

Example 5.3.2:

(1) If $\gamma=\beta=$ identity operation, fuzzy (γ,β)-open (fuzzy (γ,β)-closed) mapping coincides with fuzzy open (fuzzy closed) [11]

(2) If $\gamma=$closure operation and $\beta=$closure operation, then fuzzy (γ,β)-open (fuzzy (γ,β)-closed) mapping coincides with fuzzy θ-open (fuzzy θ-closed) mapping

(3) If $\gamma=$interior-closure operation and $\beta=$interior-closure operation, then fuzzy (γ,β)-open (fuzzy (γ,β)-closed) mapping is called fuzzy δ-open (fuzzy δ-closed) mapping.

Therom 5.3.3: Let $f:(X,T)\to(Y,T')$ be a mapping and γ and β operations on T and T' respectively.

(1) If $f(\operatorname{int}_\gamma(A))\subseteq\operatorname{int}_\beta(f(A))$ for each fuzzy set A in X, then f is fuzzy (γ,β)-open

(2) If (X,T) is γ-regular spaces, then the converse of (1) is true.

Proof: (1) Let A be any γ-open set. Then $A=\operatorname{int}_\gamma(A)$ and so $f(A)=f(\operatorname{int}_\gamma(A))$.By hypothesis, $f(A)=f(\operatorname{int}_\gamma(A))\subseteq\operatorname{int}_\beta(f(A))$. Also we have $\operatorname{int}_\beta(f(A))\subseteq f(A)$. Therefore $f(A)=\operatorname{int}_\beta(f(A))$ and hence $f(A)$ is β-open set in Y.

(2) Let (X,T) be fuzzy γ-regular space. Then we have $T=T_\gamma$. Since for each $A\in I^X$ $\text{int}_\gamma(A)$ is fuzzy open, therefore $\text{int}_\gamma(A)$ is fuzzy γ-open and by assumption, $f(\text{int}_\gamma(A))$ is fuzzy β-open set. Hence $\text{int}_\beta(f(\text{int}_\gamma(A)))=f(\text{int}_\gamma(A))$. Also $\text{int}_\gamma(A)\subseteq A$ implies $f(\text{int}_\gamma(A))\subseteq f(A)$ so that $\text{int}_\beta(f(\text{int}_\gamma(A)))\subseteq \text{int}_\beta(f(A))$.

Example 5.3.4: Let $X=\{x,y\}$ and A, B , C, D $\in I^X$ defined by

A(x) = 0.4, B (x) = 0.6, C(x) = 0.7, D = $\underline{0.6}$

A(y) = 0.3, B(y) = 0.7, C(y) = 0.6

Where $\underline{\alpha}$ denotes the constant mapping with value α.

Let $T=\{X,\varnothing A,B,\}$ and $T'=\{X,\varnothing,C,D\}$.

Then (X,T) and (X,T') are fts .

Define $\gamma:T\to I^X$ by $\gamma(X)=X,\gamma(\varnothing)=\varnothing,$

$\gamma(A)=\underline{0.4}$,

$\gamma(B)=B$ and $\beta:T'\to I^X$

by $\beta(X)=X$, $\beta(\varnothing)=\varnothing$, $\beta(C)=C$, $\beta(D)=\underline{0.5}$

Clearly (X,T) is not γ-regular spaces. Moreover $T_\gamma=\{X,\varnothing,B\}$ and $T'_\beta=\{X,\varnothing,C\}$ and so $T_\gamma^C=\{X,\varnothing,A\}$ and $T'^C_\beta=\{X,\varnothing,1-C\}$

Now consider the identity mapping $f:(X,T)\to(X,T')$ satisfying $f(x)=y$ and $f(y)=x$. Then every image of γ-closed (γ-open) is β-closed (β-open) but f is not fuzzy (γ,β)-closed.

For $B\in I^X$, we have $cl_\gamma(B)=\{(x,0.6),(y,0.9)$. So, $f(cl_\gamma(B))=\{(x,0.9),(y,0.6)\}$.

Since $f(B)=C$, we have $cl_\beta(f(B))=cl_\beta(f(C))=\underline{0.9}$.

Hence $cl_\beta(f(B)) \not\subseteq f(cl_\gamma(B))$.

Theorem 5.3.5: Suppose that f is fuzzy (γ,β)-continuous and fuzzy $(identity,\beta)$ is closed mapping then $f(A)$ is β-g-closed for each fuzzy γ-g-closed A of (X,T),

Proof: (1) Let V be any fuzzy β-open set of (Y,T') such that $f(A) \subseteq V$. Then by theorem 5.4.3, $f^{-1}(V)$ is fuzzy γ-open. Since A is γ-g-closed and $A \subseteq f^{-1}(V)$, we have $cl_\gamma(A) \subseteq f^{-1}(V)$ and hence $f(cl_\gamma(A)) \subseteq V$. Since $cl_\gamma(A)$ is closed set in (X,T) and f is (id,β) closed mapping then $f(cl_\gamma(A))$ is β-closed set of (Y,T') .Also $A \subseteq cl_\gamma(A)$ implies $f(A) \subseteq f(cl_\gamma(A))$ .This implies $cl_\beta(f(A)) \subseteq cl_\beta(f(cl_\gamma(A))) = f(cl_\gamma(A)) \subseteq V$.

Therefore $f(A)$ is β-g-closed.

Theorem 5.3.6: Suppose that $f:(X,T) \to (Y,T')$ is fuzzy (γ,β)-continuous and fuzzy $(identity,\beta)$ closed mapping. If f is injective and (Y,T') is fuzzy β-$T_{1/2}$ space, then (X,T) is γ-$T_{1/2}$.

Proof: Let A be a fuzzy γ-g-closed set of (X,T). We show that A is fuzzy γ-closed. By theorem 5.3.1 and assumptions it is obtained that $f(A)$ is β-g-closed and hence $f(A)$ is β-closed. Since f is fuzzy (γ,β)-continuous, $f^{-1}(f(A))$ is γ-closed by using theorem 5.2.3. Then it is obtained that A is fuzzy γ-closed

Theorem 5.3.7: Let $f:(X,T) \to (Y,T')$ is fuzzy (γ,β)-continuous injective mapping. If (Y,T') is fuzzy β-T_2 (resp. β-T_1), then (X,T) is γ-T_2(resp. γ-T_1).

Proof: Suppose that (Y,T') is fuzzy β-T_2. Let $p_x^\lambda, p_y^k \in$ S(X) and $p_x^\lambda \neq p_y^k$. Since f is injective, we have $f(p_x^\lambda) \neq f(p_y^k)$. As (Y,T') is fuzzy β-T_2, there exist open Q-neighbourhoods W and S of $f(p_x^\lambda)$ and $f(p_y^k)$ respectively such that $\beta(W)\overline{q}\beta(S)$. Also by fuzzy (γ,β)-continuity of f, there exist U and V of p_x^λ and p_y^k respectively such that $f(\gamma(U)) \subseteq \beta(W)$ and $f(\gamma(V)) \subseteq \beta(S)$. Then it is obtained that $f(\gamma(U))\overline{q}f(\gamma(V))$ and so $\gamma(U)\overline{q}\gamma(V)$. Thus (X,T) is fuzzy γ-T_2. The proof of the second part is similar.

Theorem 5.3.8: Let $f:(X,T) \to (Y,T')$ is fuzzy (γ,β)-continuous, injective and open mapping. If (Y,T') is fuzzy β-regular then (X,T) is γ-regular space.

***Proof*:** (1) Let $f:(X,T) \to (Y,T')$ be fuzzy (γ,β)-continuous, injective and open where (Y,T') fuzzy β-regular and γ: T $\to$ I^X and β: $T' \to I^Y$ are operation on T and T' respectively. Let $p_x^\lambda \in$ S(X) and U be an open Q-neighbourhood of p_x^λ. Since f is fuzzy open, we have f(U) is an open Q-neighbourhood of f(p_x^λ). Then by regularity of (Y,T'), we obtain $\beta(W) \subseteq f(U)$ for some open Q-neighbourhood W of f(p_x^λ). Also by (γ,β)- continuity of f, there exists an open Q-neighbourhood V of p_x^λ such that $f(\gamma(V)) \subseteq \beta(W)$. Hence $\gamma(V) = f^{-1}f(\gamma(V)) \subseteq f^{-1}(\beta(W)) \subseteq f^{-1}f(U) = U$. Thus (X,T) is γ-regular space.

5.4: Fuzzy (γ,β)-homeomorphism.

In this section we define fuzzy (γ,β)-homeomorphism, generalizing the notions of fuzzy homeomorphism, fuzzy θ-homeomorphism and fuzzy δ-homeomorphism.

Definition 5.4.1: Let γ: T $\to$ I^X be fuzzy operation on T and β: $T' \to I^Y$ be fuzzy operation on T'. A bijective mapping $f:(X,T) \to (Y,T')$ is called fuzzy (γ,β)-homeomorphism iff (i) f is (γ,β)-continuous, (ii) f^{-1} is (γ,β)-continuous.

Examples5.4.2:

(1) If $\gamma = \beta =$ identity operation, fuzzy (γ,β)-homeomorphism coincides with fuzzy open [11]

(2) If $\gamma =$ closure operation and $\beta =$ closure operation, then fuzzy (γ,β)-homeomorphism is called fuzzy θ-homeomorphism

(3) If $\gamma =$ interior-closure operation and $\beta =$ interior-closure operation, then fuzzy (γ,β)-homeomorphism is called fuzzy δ-homeomorphism.

Theorem 5.4.3: Let γ: T → I^X be fuzzy operation on T and β: $T' \to I^Y$ be fuzzy operation on T'. If $f:(X,T)\to(Y,T')$ is bijective, then the following properties of f are equivalent:

(1) f is fuzzy (γ,β)-homeomorphism

(2) f is (γ,β)-continuous and fuzzy (γ,β)-open

(3) f is (γ,β)-continuous and fuzzy (γ,β)-closed

(4) $f(\operatorname{int}_\gamma(A)) \subseteq \operatorname{int}_\beta(f(A))$ for each $A \in I^X$

Proof: $(1)\Rightarrow(2)$: Let A be γ-open set of (X,T). Since f^{-1} is (γ,β)-continuous, then by Theorem 5.1.3 we have $(f^{-1})^{-1}(A) = f(A)$ is fuzzy β-open set of (Y,T'). Consequently f is fuzzy (γ,β)-open mapping and hence $(1)\Rightarrow(2)$.

$(2)\Rightarrow(3)$: Let A be γ-closed set of (X,T).Then A^C is γ-open set of (X,T). Since $g = f^{-1}$ is (γ,β)-continuous, then by Theorem 5.1.3 we have $g^{-1}(A^C) = (g^{-1}(A))^C = (f(A))^C$ is fuzzy β-open set of (Y,T').This implies $f(A)$ is fuzzy β-closed set of (Y,T'). Consequently f is fuzzy (γ,β)-closed mapping and hence $(2)\Rightarrow(3)$.