CHAPTER 3:

γ-operations and its related topoics in Fuzzy topological spaces

3.1 Introduction:

It is known that S. Kasahara [51] introduced the notions of γ-continuous mapping, γ-compact and γ- closed graphs by introducing an operation γ on a topology. After kasahara, D.S. Jankovic [4] defined the concept of γ- closure, γ-closed (open) functions and studied some properties of functions with γ-closed graphs. Then H. Ogata [5] defined the notion γ-open sets and used it to investigate some new separation axioms γ-T_i = 0, 1/2, 1. Moreover Ogata introduced different (γ, β) -function and established some properties of this notions. In 1992, F. U. Rehman and B. Ahmad [4, 91] defined and investigated several properties of γ-interior, γ-exterior, γ-closure and γ-boundary points in Topological Spaces and studied the characterizations of (γ, β)-continuous mappings. Thereafter B. Ahmad and S. Hussain continued studing different topological concepts and properties relate to γ-operations on topological space. So far, no attempt has been made to relate the above concepts to fuzzy topological spaces. In first section, we introduce and study the concepts of an operation γ on a fuzzy topology T on a set X. Then we develop the notions of fuzzy γ-open and investigate some properties of these notions. We show that under a certain condition, the family of all of fuzzy γ-open sets forms a fuzzy topology on X.

In next section, we define two different fuzzy γ-closure and discuss the relation between them. Moreover we show that they are equivalent under some suitable conditions.

The notions of γ-continuous and its some properties are introduced and studied in section 4. Moreover we obtain some basic properties of these concepts.

In the last section we develop the concepts of fuzzy γ-compactness. Fuzzy filterbases are then used to characterize this concept. Also some expected basic properties γ-compactness are explored. Two papers on results of this chapter are published in journal of fuzzy Mathematics [26, 27].

3.2. Fuzzy γ-operations and Fuzzy γ-open set:

Definition 3.2.1: Let (X, T) be a fuzzy topological space. An operation γ on T is a mapping from T into I^X such that U $\subseteq$ γ(U) for every U$\in T$. The Mapping $\gamma : T \to I^X$ defined by

(1) γ(U) = U for every U $\in$ T is an operation on T and this is called identity operation;

(2) γ(U) = Cl(U) for every U $\in$ T is an operation on T and this is called closure operation;

(3) γ(U) = Int(Cl(U) for every U $\in$T is an operation on T and this is called interior-closure operation

Definition 3.2.2: Let (X, T) be a fts. A fuzzy subset A of X will be called a fuzzy γ -open iff $\forall p_x^\lambda qA,$ there exists an open Q -neighborhood U of p_x^λ such that $\gamma(U) \subseteq A$.

T_γ denotes the set of all fuzzy γ -open sets.

Definition 3.2.3: Let $X = \{x, y\}$ and A, B, C $\in I^X$ defined by

A = $\underline{0.6}$, B (x) = 0.6, B (y) = 0.7, C = $\underline{0.3}$,

Where $\underline{\alpha}$ denotes the constant mapping with value α. Let $T = \{X ,\varnothing A, B, C\}$

Then (X,T) is a fts. Define $\gamma : T \to I^X$ by $\gamma(X) = X, \gamma(\varnothing) = \varnothing$ $\gamma(A) = A, \gamma(B) = B, \gamma(C) = \underline{0.5}$. Then we can easily see that $T_\gamma = \{X, \varnothing, A, B\}$. Thus X, ∅, A, B are fuzzy γ-open but C is not fuzzy γ-open set.

Remark 3.2.4: $T_\gamma \subseteq T$.

Proof: Let $A \in T_\gamma$ and $p_x^\lambda qA$. Then there exists an open Q -neighborhood U of p_x^λ such that $\gamma(U) \subseteq A$ .By definition of γ, we have $U \subseteq \gamma(U) \subseteq A$ which implies that A is open.

Definition 3.2.5: Let (X,T) be a fts and γ be an operation on T and $p_x^\lambda \in S(X)$. The γ-interior of a fuzzy set $A \in I^X$ is denoted by $Int_\gamma(A)$ and defined as $p_x^\lambda q \operatorname{int}_\gamma(A)$ if there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U) \subseteq A$.

Theorem 3.2.6: Let (X,T) be a fts and γ be an operation on T. Then for $A \in I^X$,

(1) $Int_\gamma(A) \subseteq A$

(2) $Int_\gamma(A)$ is a fuzzy open set.

(3) A is fuzzy γ-open iff $Int_\gamma(A) = A$

Proof: (1) Let $p_x^\lambda q \operatorname{int}_\gamma(A)$. Then there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U) \subseteq A$ .By definition of γ, we have $U \subseteq \gamma(U) \subseteq A$. Since $p_x^\lambda qU$, we have $p_x^\lambda qA$ Thus $p_x^\lambda q \operatorname{int}_\gamma(A) \Rightarrow p_x^\lambda qA$. This means $Int_\gamma(A) \subseteq A$.

(2) It is obvious

(3) Let A be fuzzy γ-open and $p_x^\lambda qA$. Then there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U) \subseteq A$ .This shows that $p_x^\lambda q \operatorname{int}_\gamma(A)$. Thus $p_x^\lambda qA \Rightarrow p_x^\lambda q \operatorname{int}_\gamma(A)$. Hence $A \subseteq Int_\gamma(A)$ Since by (1), $Int_\gamma(A) \subseteq A$ we get $Int_\gamma(A) = A$.

Conversely let $Int_\gamma(A) = A$. We want to prove that A is fuzzy γ-open.

Let $p_x^\lambda qA = Int_\gamma(A)$. Then there exists an open Q-neighbourhood U of p_x^λ such that $\gamma(U) \subseteq A$ .Consequently A is fuzzy γ-open.

Definition 3.2.7: An operation γ on fuzzy topology T is said to be fuzzy regular if for every open Q-neighborhoods U and V of each p_x^λ, there exists an open Q-neighborhood W of p_x^λ such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Example 3.2.8:

(1) In the Example 3.2.3, the operation γ is regular.

(2) Let X = {x, y} and consider the following fuzzy topology,

$T = \{X, \varnothing\} \bigcup \{p_x^\lambda : \lambda \geq .5\} \bigcup \{p_x^\lambda \cup p_y^{.5} : \lambda \geq .5\}$.

Let $\gamma : T \to I^X$ be defined by $\gamma(U) = X$ if $U = p_x^{.5}$ and $\gamma(U) = U$ if $U \neq p_x^{.5}$.

Then γ is not regular operation on T. Indeed, for $p_x^\lambda, \lambda > 0.5$, we have $p_x^{0.5}, (p_x^{0.5} \cup p_y^{0.5}) \in N^Q(p_x^\lambda)$ and $\gamma(p_x^{0.5}) \bigcap \gamma(p_x^{0.5} \cup p_y^{0.5}) = p_x^{0.5}$, but there is no $U \in N^Q(p_x^\lambda)$ such that $\gamma(U) \subseteq \gamma(p_x^{0.5}) \bigcap \gamma(p_x^{0.5} \cup p_y^{0.5})$

Definition 3.2.9: A fuzzy operation γ on T is said to be fuzzy open if for every open Q-neighborhood U of p_x^λ, there exists a fuzzy γ-open set A such that p_x^λ q A and A $\subseteq$ γ(U).

Theorem 3.2.10: Let A be a fuzzy subset of (X,T). If A is a fuzzy γ-open set then A is open.

***Proof*:** Since $T_\gamma \subseteq$ T, the result follows immediately.

Theorem 3.2.11: If A_j is fuzzy γ-open set for every j$\in$ J then $\bigcup \{A_j \mid j \in J\}$ is fuzzy γ-open.

***Proof*:** Let B = $\bigcup \{A_j \mid j \in J\}$ and p_x^λ q B. Then there exists some $A_j \in$ T such that p_x^λ q A_j.. Since A_j is fuzzy γ-open set, so there exists an open Q-neighbourhood W of p_x^λ such that γ(W) $\subseteq$ A_j and so. $\gamma(W) \subseteq B$. Thus B is fuzzy γ-open.

Theorem 3.2.12: Let γ be a fuzzy regular operation on T.

(i). If A and B are fuzzy γ-open set then A $\bigcap$ B is γ-open set.

(ii). T_γ is fuzzy topology on X.

***Proof*:** (i) Let p_x^λ q (A $\cap$ B). Then p_x^λ q A and p_x^λ q B. Since A and B are fuzzy γ-open sets, there exists open Q-neighborhoods U, V of p_x^λ such that γ(U) $\subseteq$ A and γ(V) $\subseteq$ B. Again since γ is fuzzy regular operation, there exists an open Q-neighborhood W of p_x^λ such that γ(W) $\subseteq$γ(U)$\cap$γ(V). Then we have γ(W)(x) ≤ min{γ(U)(x), γ(V)(x)}. Consequently γ(W)(x) ≤ min{A(x), B(x)}.Thus we get γ(W) $\subseteq$ A $\cap$ B which shows that A $\cap$ B is fuzzy γ-open set.

(ii) X and ∅ are fuzzy γ-open sets together with (i) and theorem 3.2.11 that T_γ is fuzzy topology on X.

Definition 3.2.13: A fuzzy topological (X,T) is called fuzzy γ-regular space if for each fuzzy point $p_x^\lambda \in$ S(X) and every open Q-neighborhood V of p_x^λ, there exists an open Q-neighborhood W of p_x^λ such that γ(W) $\subseteq$ V.

Examples 3.2.14:

(1) For closure operation γ, fuzzy γ-regular space coincides with fuzzy regular [64]

(2) For Interior-closure operation γ, fuzzy γ-regular space coincides with fuzzy semi-regular [64]

(3) For closure operation γ, fuzzy γ-regular space coincides with fuzzy almost-regular. [64]

Theorem 3.2.15: (X,T) is fuzzy γ-regular space if and only if $T = T_\gamma$.

***Proof*:** (Necessity): It suffices to prove that $T \subseteq T_\gamma$. Let A be a fuzzy open set and p_x^λ q A. Then A(x) > 1-λ for some x$\in X$ .and this shows that A is open q-neighborhood of p_x^λ. Since (X,T) is fuzzy γ-regular space, there exists an open q-neighborhood W of p_x^λ such that γ(W) $\subseteq$A and hence A is fuzzy γ-open set.

(Sufficiency): Let p_x^λ be a fuzzy point and V be an open q-neighborhood of p_x^λ. Since T = T_γ, V is fuzzy γ-open set. Therefore there is an open q-neighborhood W of p_x^λ such that γ(W) ⊆ V. This shows that (X,T) is a fuzzy γ-regular space.

Example 3.2.16: Let $X = \{x, y\}$ and A, B, C, D ∈ I^X defined by

A(x) = 0.4 , A(y) = 0.3, B (x) = 0.6, B(y) = 0.7,

Let $T = \{X, \varnothing A, B,\}$ Then (X,T) is fts. Define $\gamma : T \to I^X$ by $\gamma(X) = X, \gamma(\varnothing) = \varnothing$, $\gamma(A) = \underline{0.4}$, $\gamma(B) = B$.

Clearly (X, T) is not a γ-regular spaces. Moreover $T_\gamma = \{X, \varnothing, B\}$

Example 3.2.17: Let $X = \{x, y\}$ and A, B , C, D ∈ I^X defined by

A(x) = 0.5, B(x) = 0.5, C(x) = 0.4, D = $\underline{0.4}$

A(y) = 0.6, B(y) = 0.4, C(y) = 0.6

where $\underline{\alpha}$ denotes the constant mapping with value α. Let $T = \{X, \phi, A, B, C, E\}$. Then (X,T) is fts. Define $\gamma : T \to I^X$ by $\gamma(X) = X, \gamma(\varnothing) = \varnothing$, $\gamma(A) = \underline{0.7}$, $\gamma(B) = B$, $\gamma(C) = C$, $\gamma(D) = D$. Clearly (X,T) is γ-regular space and $T_\gamma = \{X, \phi, A, B, C, D\}$

3.3 Fuzzy γ-closures and its properties:

In this section we introduce two different types of γ-closure - T_γ-Cl(A) and Cl_γ(A) containing a fuzzy set A of (X,T) and investigate relation between them.

Definition 3.3.1: A fuzzy subset A of (X,T) is said to be fuzzy γ-closed set if its complement A^c is fuzzy γ-open.

Definition 3.3.2: For a fuzzy subset A of (X,T) and T_γ, we define T_γ-Cl(A) as follows

$$T_\gamma\text{-Cl}(A) = \inf \{ F : A \subseteq F, F^c \in T_\gamma \}$$

Theorem 3.3.3: For a fuzzy point p_x^λ in X, $p_x^\lambda \in T_\gamma$-Cl(A) if and only if

V q A for any V∈ T_γ such that p_x^λ q V.

***Proof*:** We have $p_x^\lambda \in T_\gamma$-Cl(A) if and only if for every fuzzy γ-closed set

F ⊇ A, $p_x^\lambda \in$F or F(x) ≥ λ. By taking complement, this fact can be stated as follows:

$p_x^\lambda \in T_\gamma$-Cl(A)if and only if for every fuzzy γ-open set V ⊆ A^c, V(x) ≤ 1-λ .

In other words, $p_x^\lambda \in T_\gamma$-Cl(A) if and only if for every fuzzy γ-open set V satisfying V(x) > 1-λ , V is not contained A^c. Therefore $p_x^\lambda \in T_\gamma$-Cl(A) if and only if for every fuzzy γ-open set V satisfying V(x) > 1-λ and V is quasi-coincident with A. Thus we have proved that $p_x^\lambda \in T_\gamma$-Cl(A) if and only if V q A for every fuzzy γ-open set V such that p_x^λ q V.

Theorem 3.3.4 Let A and B be fuzzy subsets of (X,τ).

i. A is γ-closed if and only T_γ-Cl(A) = A.

ii. A ⊆ T_γ-Cl(A).

iii. If A⊆ B then T_γ-Cl(A) ⊆ T_γ-Cl(B)

iv. T_γ-Cl(A) is fuzzy γ-closed set .

***Proof*:** (i) (Necessity): Let A be fuzzy γ-closed set, then by definition 3.3.2. T_γ-Cl(A) = A.

(Sufficiency): Let T_γ-Cl(A) = A. Then we want prove that A^c is fuzzy γ-open set. Let p_x^λ q A^c . This means $p_x^\lambda \notin$ A = T_γ-Cl(A). Then there exists a fuzzy γ-open set V and p_x^λ q V such that V is not quasi-concident with A and so V ⊆A^c. Since V is fuzzy γ-open, so for p_x^λ q V, there exists an open Q-neighbourhood W and such that γ(W) ⊆V. Hence γ(W) ⊆ A^c that shows A^c is fuzzy γ-open set. Consequently A is fuzzy γ-closed set.

(ii) It is obvious.

(iii) Let $p_x^\lambda \in T_\gamma$-Cl(A) . Let V be fuzzy γ-open set and p_x^λ q V. Then we have V q A. Since A ⊆ B, so V q B. This shows $p_x^\lambda \in T_\gamma$-Cl(B). Thus T_γ-Cl(A) ⊆ T_γ-Cl(B)

(iv) Here we want to prove that $T_\gamma\text{-}Cl(T_\gamma\text{-}Cl(A)) = T_\gamma\text{-}Cl(A))$. Let G = $T_\gamma\text{-}Cl(T_\gamma\text{-}cl(A))$ and H = $T_\gamma\text{-}Cl(A))$ .Let $p_x^\lambda \in T_\gamma\text{-}Cl(T_\gamma\text{-}Cl(A))$ and V be fuzzy γ-open set and p_x^λ q V. Then we have V q H which implies V(y) +H(y) > 1 for some y∈ X .. Let H(y) = r, r∈ [0,1]. Then $p_y^r \in$ H = $T_\gamma\text{-}cl(A))$ and V is fuzzy γ-open set and p_y^r q V. Hence by theorem 3.3.3 we get V q A. This shows that $p_x^\lambda \in T_\gamma\text{-}Cl(A))$. Again, let

$p_x^\lambda \in T_\gamma$-Cl(A). Then by (ii), $p_x^\lambda \in T_\gamma\text{-}Cl(T_\gamma\text{-}Cl(A))$ Thus we have shown that

$p_x^\lambda \in T_\gamma\text{-}Cl(T_\gamma\text{-}Cl(A)) \Leftrightarrow p_x^\lambda \in T_\gamma$-Cl(A). Hence $T_\gamma\text{-}Cl(T_\gamma\text{-}Cl(A)) = T_\gamma$-Cl(A) and by (i) T_γ-Cl(A) is fuzzy γ-closed set.

Now we introduce the following definition of Cl_γ(A).

Definition 3.3.5: A fuzzy point $p_x^\lambda \in S(X)$ is in the fuzzy γ-closure of fuzzy set A of X i.e. in Cl_γ(A) if γ(V) q A for each open Q-neighborhood V of p_x^λ.

Theorem 3.3.6: For a fuzzy subset A of (X,T) the following properties hold.

(i) A ⊆ Cl(A) ⊆ $Cl_\gamma(A)$ ⊆ T_γ-Cl(A)

(ii) If (X,T) is fuzzy γ-regular space then Cl(A) = $Cl_\gamma(A)$ = T_γ-Cl(A)

(iii) $Cl_\gamma(A)$ is fuzzy closed subset of (X,T).

***Proof*:** (i) Let $p_x^\lambda \in$ Cl(A). Let V be an open Q-neighborhood of p_x^λ. Then V q A and by the definition of γ, we get γ(V) q A. This shows that $p_x^\lambda \in Cl_\gamma$(A). Therefore Cl(A) ⊆ Cl_γ(A).

Again let $p_x^\lambda \notin T_\gamma$-Cl(A). Then there exists a fuzzy γ-open set V such that p_x^λ q V and V is not quasi-concident with A. Then we have V(x) + A(x) ≤ 1 for all $x \in X$. But V is fuzzy γ-open set, so there exists an open Q-neighbourhood W of p_x^λ such that γ(W) $\subseteq$ V. Then γ(W) is not quasi-concident with A. Since W is open q-neighborhood of p_x^λ, then by definition ,we have $p_x^\lambda \notin$ Cl_γ(A).Hence Cl_γ(A) $\subseteq$ T_γ-Cl(A).Thus we have $A \subseteq Cl(A) \subseteq Cl_\gamma(A) \subseteq T_\gamma$-Cl(A)

(ii) By theorem 3.2.15, we have $T = T_\gamma$ and hence Cl(A) = T_γ-Cl(A). By using (i) it is shown that Cl(A) = Cl_γ(A) = T_γ-Cl(A)

(iii) Here we want to prove that Cl(Cl_γ(A)) = Cl_γ(A). Let $p_x^\lambda \in$ Cl(Cl_γ(A)) and V be an open Q-neighborhood of p_x^λ.Then we have V q Cl_γ(A) which implies V(y) + Cl_γ(A)(y) >1 for some $y \in X$ .Let $Cl_\gamma(A)(y)$ = r, $r \in [0,1]$. Then $p_y^r \in$ $Cl_\gamma(A)$ and V is a fuzzy open Q-neighborhood of p_y^r. By definition, we get γ(V) q A. This shows $p_x^\lambda \in$ $Cl_\gamma(A)$. Again for $p_x^\lambda \in$ Cl(A). Then by (i) we have $p_x^\lambda \in$ $Cl_\gamma(A)$.Thus we have shown that

$p_x^\lambda \in$ $Cl_\gamma(A) \Leftrightarrow$ $p_x^\lambda \in$ Cl(A). Hence Cl($Cl_\gamma(A)$)= $Cl_\gamma(A)$.

Theorem 3.3.7: Let A be a fuzzy subset of (X,T).

i. A is fuzzy γ-closed if and only if Cl_γ(A) = A.

ii. T_γ-Cl(A) =A if and only if Cl_γ(A) =A.

iii. A is fuzzy γ-open if and only if Cl_γ(A^c) = A^c

iv. T_γ-Cl(Cl_γ(A)) = T_γ-Cl(A) = Cl_γ(T_γ-Cl(A))

v. If $A \subseteq$ B then Cl_γ(A) $\subseteq$ Cl_γ(B).

***Proof*:** (i) (Necessity): we prove that Cl$_\gamma$(A) $\subseteq$ A. Let $p_x^\lambda \notin$ A. Then p_x^λ q A^c. Since A^c is fuzzy γ-open, there exists an open Q-neighbourhood V of p_x^λ such that γ(V) $\subseteq$ A^c which implies that γ(V) is not quasi-concident with A. Since V is open Q-neighborhood of p_x^λ, it shows that $p_x^\lambda \notin$ Cl$_\gamma$(A). Hence Cl$_\gamma$(A) $\subseteq$ A. Again by theorem 3.3.6, we have A $\subseteq$ Cl$_\gamma$(A). Thus Cl$_\gamma$(A) = A.

(Sufficiency): We want to prove that A^c is fuzzy γ-open set. Let p_x^λ q A^c. This means $p_x^\lambda \notin$ A = Cl$_\gamma$(A). Then there exists a fuzzy open Q-neighborhood V of p_x^λ such that γ(V) $\bar{q}A$. Therefore γ(V) $\subseteq A^c$ so that A^c is fuzzy γ-open set and hence A is fuzzy γ-closed set.

(ii) follows from (i) and theorem 3.3.4 (i).

(iii) follows from (i) and definition 3.3..1.

(iv) By the theorem 3.3.4 (iv), we have T_γ-Cl(A) is fuzzy γ-closed subset of X. Then by (i) we get T_γ-Cl(A) = Cl$_\gamma$(T_γ-Cl(A)). Again by theorem 3.3.6(i), we have A $\subseteq$ Cl(A) $\subseteq$ Cl$_\gamma$(A) $\subseteq T_\gamma$-Cl(A). Also by thorem 3.3.4 (iii) we get T_γ-Cl(A) $\subseteq$ T_γ-Cl(Cl$_\gamma$(A)). Hence we can obtain Cl$_\gamma$(A) T_γ-Cl(A) $\subseteq$ T_γ-Cl(Cl$_\gamma$(A)). By using these inclusions and theorem 3.3.4 (iii), we can find

T_γ-Cl (Cl$_\gamma$(A)) $\subseteq T_\gamma$-Cl(T_γ-Cl(A)) $\subseteq$ T_γ-Cl(T_γ-Cl(Cl$_\gamma$(A))). Since T_γ-Cl(A) is fuzzy γ-closed, so we get T_γ-Cl (Cl$_\gamma$(A)) $\subseteq$ (T_γ-Cl(A)) $\subseteq$ T_γ-Cl(Cl$_\gamma$(A)). Thus T_γ-Cl(A) = T_γ-Cl (Cl$_\gamma$(A)) and hence T_γ-Cl (Cl$_\gamma$(A)) = T_γ-Cl(A) = Cl$_\gamma$(T_γ-Cl(A))

(v) Let $p_x^\lambda \in$ Cl$_\gamma$(A). Let V be fuzzy open Q-neighborhood of p_x^λ. Then we have γ(V) q A. Since A$\subseteq$ B so we get γ(V) q B and so $p_x^\lambda \in$ Cl$_\gamma$(B). Thus Cl$_\gamma$(A) $\subseteq$ Cl$_\gamma$(B).

Theorem 3.3.8: If γ is fuzzy open operation and A is fuzzy subset of X then

i. $Cl_\gamma(A) = T_\gamma$-Cl(A).

ii. $Cl_\gamma(Cl_\gamma(A)) = Cl_\gamma(A)$ i.e. $Cl_\gamma(A)$ is γ-closed subset of (X,τ).

***Proof*:** (i) Suppose $p_x^\lambda \notin Cl_\gamma(A)$. Then there exists a fuzzy open q-neighborhood V of p_x^λ such that $\gamma(V)\ \bar{q}A$. Since γ is fuzzy open operation, there exists a fuzzy γ-open set W and p_x^λ q W such that $W \subseteq \gamma(V)$. Therefore W $\bar{q}A$ and so $p_x^\lambda \notin T_\gamma$-Cl(A).

Hence T_γ-Cl(A) $\subseteq$ $Cl_\gamma(A)$.Again by theorem 3.3.6(i) we have $Cl_\gamma(A) \subseteq T_\gamma$-Cl(A). Therefore $Cl_\gamma(A) = T_\gamma$-Cl(A).

(ii) By (i) and theorem 3.3.4 (iv) we have

$Cl_\gamma(Cl_\gamma(A)) = T_\gamma$-Cl($Cl_\gamma(A)$) = T_γ-Cl(T_γ-Cl(A))= T_γ-Cl(A) = $Cl_\gamma(A)$.

3.4. Fuzzy γ-Continuous Mapping:

In this section fuzzy γ-continuity is introduced and studied in the light of the notions of q-coincidence and fuzzy points which generalize the some types of continuity functions as in [21, 38, 64]

Definition 3.4.1: A fuzzy point p_x^λ is called a fuzzy γ-cluster point of a fuzzy set A iff $\gamma(U)$ q A for every open Q-neighbouhood U of p_x^λ. $cl_\gamma(A)$ is union of all γ-cluster points of A.

Examples 3.4.2:

(1) If γ is closure operation then the γ-cluster point coincides with the θ-cluster point

(2) If γ is interior-closure operation then the γ-cluster point implies δ-cluster point

Definition 3.4.3: Let (X,T) and (Y,T') be fuzzy topological spaces, and

γ be a fuzzy operation on T'. A mapping f of X into Y is said to be fuzzy γ-continuous if for each fuzzy point p_x^λ in X and for each open Q-neighbourhood V of $f\,(p_x^\lambda)$ in Y, there exists an open Q-nighbourhood. U of p_x^λ such that f(U) $\subseteq \gamma(V)$.

Examples3.4.4:

(1) If γ is the identity operation on T', then the γ-continuity coincides with the fuzzy continuity.[21,83]

(2) If γ is the closure operation on T', then the γ-continuity coincides with the fuzzy weakly θ-continuous. [64]

(3) If γ is the interior-closure operation, then the γ-continuity coincides with the fuzzy almost continuity [38]

Theorem 3.4.5: Following properties (i), (ii), (iii), (iv) and (v) of a fuzzy mapping $f:(X,T)\to(Y,T')$ satisfy the implication rules (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v)

(i) $f:(X,T)\to(Y,T')$ is fuzzy γ-continuous mapping.

(ii) $f(cl(A)) \subseteq cl_\gamma(f(A)))$ for every $A\in I^X$

(iii) $cl(f^{-1}(B)) \subseteq f^{-1}(cl_\gamma(B))$ for every $B\in I^Y$

(iv) for every fuzzy γ-closed set B of Y, $f^{-1}(B)$ is closed set in X.

(v) for every γ-open set B of Y, $f^{-1}(B)$ is open set in X .

Proof: (i) ⇒ (ii). Let $p_x^\lambda \in cl(A)$. Let V be any open Q-nbd. of $f(p_x^\lambda)$.Then there exists an open Q-nbd. of U of p_x^λ such that f(U) $\subseteq \gamma(V)$. Since $p_x^\lambda \in cl(A)$, then U q A. This implies, f(U) q f(A) ⇒ $\gamma(V)$ q f(A) ⇒ $f(p_x^\lambda) \in cl_\gamma(f(A)) \Rightarrow p_x^\lambda \in f^{-1}(cl_\gamma(A)))$.

Thus $cl(A) \subseteq f^{-1}(cl_\gamma(f(A)))$. That $f(cl(A)) \subseteq cl_\gamma(f(A))$

(ii) ⇒ (iii). By (ii) we have $f(cl(f^{-1}(B))) \subseteq cl_\gamma(ff^{-1}(B)) \subseteq cl_\gamma(B)$ and so

$cl(f^{-1}(B)) \subseteq f^{-1}(cl_\gamma(B))$

(iii)⇒ (iv). Let B be a fuzzy γ-closed set in Y. Then $cl_{\gamma}(B) = B$. By (iii)

$cl(f^{-1}(B)) \subseteq f^{-1}(cl_{\gamma}(B)) = f^{-1}(B)$ which implies that $cl(f^{-1}(B)) = f^{-1}(B)$.Thus

$f^{-1}(B)$ is fuzzy closed set in X.

(iv) ⇒ (v) Let B be fuzzy γ-open set in Y. Then B^c is fuzzy γ-closed set in Y.

Then by (iv) $f^{-1}(B^c)$ is fuzzy closed set in X. But $f^{-1}(B^c) = (f^{-1}(B))^c$ and hence

$f^{-1}(B)$ is fuzzy open set in X.

Corollary 3.4.6: If Y is γ-regular space then all properties of theorem 3.6 are equivalent

Proof: By Theorem 3.4.6, we have (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v), so it is sufficient to prove (v) ⇒ (i). Let p_x^{λ} be a fuzzy point in X and V be a fuzzy open Q-neighborhood of f(p_x^{λ}). Since Y is fuzzy γ-regular space, then V is fuzzy γ-open set in Y. Therefore by assumption $f^{-1}(V)$ is fuzzy open set in X. Also f(p_x^{λ}) q V ⇒ p_x^{λ} q $f^{-1}(V)$. Thus

$U = f^{-1}(V)$ is an open Q-nbd. of p_x^{λ} and $f(U) = f(f^{-1}(V)) \subseteq V \subseteq \gamma(V)$. Hence f is fuzzy γ-continuous mapping.

Corollary 3.4.7: If γ is open operation, then all the properties of theorem 3.4.6 are equivalent.

***Proof*:** By Theorem 3.4.6 (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v), so it is sufficient to prove that

(v) ⇒ (i). Let p_x^{λ} be a fuzzy point in X and V be a fuzzy open q-neighborhood of f(p_x^{λ}).

Since γ is open operation, there exists a fuzzy γ-open set A and $f(p_x^{\lambda})$ q A such that

A ⊆ γ(V). Again since A is fuzzy γ-open set in Y, by our assumption, $f^{-1}(A)$ is fuzzy open set in X. Also p_x^{λ} q $f^{-1}(A)$ so that $U = f^{-1}(A)$ is an open Q-neighbourhood of p_x^{λ} and $f(U) = f(f^{-1}(A)) \subseteq A \subseteq \gamma(V)$ .This shows that f is fuzzy γ-continuous mapping.

3.5 Fuzzy γ-compactness :

In this section, the notion of γ-compactness in fuzzy setting is defined and developed theory in this field. Then we have developed notions of fuzzy γ-convergence and γ-accumulation of fuzzy filterbase and applied to characterize fuzzy γ-compact.

Definition 3.5.1: Let (X,T) be a fuzzy topological space, and γ be an operation on T. A fuzzy filter base F in X is said to γ-accumulates to p_x^λ in X if $\gamma(V)$ q A for every A $\in$ F and every open Q-neighbourhood V of p_x^λ

Definition 3.5.2: A fuzzy filter base F in X is said to γ-convergent to p_x^λ if for every open Q-neighbourhood V of p_x^λ, there exists an A $\in$ F such that $F \subseteq \gamma(V)$ and $p_x^\lambda \in cl_\gamma(B)$ for every B $\in$ F.

Theorem 3.5.3: A fuzzy point p_x^λ in X is fuzzy γ-cluster point of a filter base F iff $p_x^\lambda \in cl_\gamma(B)$ for each B $\in$ F.

***Proof*:** It is straightforward. In facts it follows from the definition 3.5.1.

Theorem 3.5.4: Let (X,T) be a fuzzy topological space and γ be an operation on T. Then the followings hold:

(1) If a fuzzy filterbase F in X γ-converges to p_x^λ in X, then F γ-accumulates to p_x^λ

(2) If a fuzzy filterbase F in X is contained in a filterbase which γ-accumulate to p_x^λ in X, then F γ-accumulate to p_x^λ

(3) If a maximal filterbase in X γ-accumulates to p_x^λ, then it γ-converges to p_x^λ

***Proof*:**

(1) It follows from definition 3.5.2

(2) Let H be a filter base such that $F \subseteq H$ and it γ-accumulate to p_x^λ. Then we have $\gamma(V)$ q A for every A ∈ H and every open Q-neighbourhood V of p_x^λ. Since $F \subseteq H$, then $\gamma(V)$ q B for every B ∈ F. This implies F also γ-accumulate to p_x^λ.

(3) follows obiously.

Theorem 3.5.5: Let (X,T) be a fuzzy topological space and γ be an operation on T. If a filterbase F in X γ-accumulates to p_x^λ, then there exists a fuzzy filterbase H in X such that $F \subseteq H$ and H γ-converges to p_x^λ.

Proof: Let the filterbase F γ-accumulate to p_x^λ .Then $p_x^\lambda \in cl_\gamma(A)$, for every A∈ F. Hence for every open Q-nbd. U of p_x^λ and for each A ∈ F, $\gamma(U)qA$ which implies that $\gamma(U) \bigcap A \neq 0_x$. Consider the set G = { γ(U) ⋂ A : A ∈ F and p_x^λ q U ∈ T }. Let $G_1, G_2 \in G$. Then $G_1 \bigcap G_2 = (A_1 \bigcap A_2) \bigcap (\gamma(U_1) \bigcap \gamma(U_2)) = A \bigcap (\gamma(U_1) \bigcap \gamma(U_2))$ for every $A \in F$ and $U_1, U_2 \in N^Q(p_x^\lambda)$. Since γ is regular, then there exists an open Q-nbd. W of p_x^λ such that $\gamma(W) \subseteq \gamma(U_1) \bigcap \gamma(U_2)$. Hence $G_1 \bigcap G_2 \supseteq \gamma(W) \bigcap A$. Let $G_3 = \gamma(W) \bigcap A \in F$. Then we have $G_3 \subseteq G_1 \bigcap G_2$ and hence G is fuzzy filter base in X. Now the fuzzy set $H = \{B \in I^X : C \subseteq B$ for some $C \in G\}$ is fuzzy filter generated by G and γ-converges to p_x^λ and F⊆ H.

Definition 3.5.6: Let (X,T) be fts and γ be a fuzzy operation on T. Then (X,T) is said to be fuzzy γ-compact if for every open cover $\{U_i : i \in \Lambda\}$ of X, there exists a finite subset Λ_0 of Λ such that $\bigcup\{\gamma(U_i) : i \in \Lambda_0\} = 1_X$

Examples 3.5.7:

(1) If γ is identity operation, then fuzzy γ-compactness reduces to fuzzy compact [21].

(2) If γ is closure operation, then fuzzy γ-compactness reduces to fuzzy almost compact [24]

(3) If γ is the interior-closure operation, then fuzzy γ-compactness coincides with fuzzy nearly compact [43]

Theorem 3.5.8: Every fuzzy compact space is fuzzy γ-compact.

Proof: Let $\{V_\alpha : \alpha \in \Lambda\}$ be fuzzy open cover of X. Since X is fuzzy compact, there is a finite subset Λ_0 of Λ such that $\bigcup\{U_\alpha : \alpha \in \Lambda_0\} = 1_X$ .Then by definition of γ, $\bigcup\{\gamma(U_\alpha) : \alpha \in \Lambda_0\} = 1_X$ Thus X is fuzzy γ-compact.

Remark 3.5.9: The converse of above theorem is not true as the example of an almost compact but not compact space given by Di Concilio and Gerla [24].

Theorem 3.5.10: If fts (X,T) is fuzzy γ-compact for some operation γ on T such that (X,T) is fuzzy γ-regular, then (X,T) is fuzzy compact.

Proof: Let $C = \{U_i : i \in \Lambda\}$ be an open cover of X. Since X is fuzzy γ-regular, then for each $i \in \Lambda$, $\gamma(V_i) \subseteq U_i$. Since V_i is open set, therefore the set $\{V_i : i \in \Lambda\}$ is an open cover of X. Since (X,T) is fuzzy γ-compact, there is a finite subset Λ_0 of Λ such that $\bigcup\{\gamma(V_i) : i \in \Lambda_0) = 1_X$. Therefore $\bigcup\{U_i : i \in \Lambda_0) = 1_X$ and so X is fuzzy compact.

Theorem 3.5.11: Let (X,T) be fts and γ be a fuzzy operation on T. Then the following conditions are equivalent:

(1) (X,T) is fuzzy γ-compact

(2) Each fuzzy filterbase in X γ-accumulates to some fuzzy point of X.

(3) Each fuzzy maximal filterbase in X γ-converges to some fuzzy point of X.

Proof: (1) ⇒ (2). Suppose that there exists a fuzzy filterbase F in X which does not γ-accumulate to any fuzzy point p_x^λ of X. Then there exists $F_x \in F$ and an open Q-nbd. of p_x^λ such that F_x is not quasi-coincident with $\gamma(V_x)$. Since X is fuzzy γ-compact and the family $C = \{V_x : x \in X\}$ is open cover of X, there exists a finite subfamily $\{V_{x_i} : i = 1,2,\cdots n\}$ of C such that $\bigcup_{i=1}^{n} \gamma(V_{x_i}) = 1_X$ But F is filter base, so there is a subfamily $\{F_{x_i} : i = 1,2,\cdots n\}$ of F such that $\bigcap_{i=1}^{n} F_{x_i} \neq 0_x$ and consequently $F_{x_m} q\gamma(V_{x_m})$ for some $m \in$ {1,2, …,n} which is a contradiction. Hence (1) ⇒ (2).

(2) ⇒ (3). Let G be fuzzy maximal filterbase in X and F be any fuzzy fliltebase in X which γ-accumulates to some fuzzy point of X, say p_x^{λ}. Then G is also γ-accumulates to p_x^{λ} and by the 3.5.4(3), G γ-converges to p_x^{λ}

(3) ⇒ (2). Since each filterbase is contained in a maximal filterbase in X, (3) obiviously implies (2)

(2) ⇒ (1) Let $C = \{U_i : i \in \Lambda\}$ be an open cover of X such that $\bigcup\{\gamma(U_i) : i \in \Lambda_0\} \neq 1_X$, where Λ_0 is finite subset of Λ. Let D denote the set of all sets of the form $\bigcap\{(\gamma(U_i))^c : i \in \Lambda_0\}$. Since $\bigcap(\gamma(U_i))^c \neq 0_X$, D is fuzzy filterbase in X. and so by our assumption, it fuzzy γ-accumulates to some point p_x^{λ} in X. But p_x^{λ} is quasi-coincident with some $U \in$ C and so $(\gamma(U))^c \in D$. Thus U is an open Q-nbd. of p_x^{λ} such that $\gamma(U)(x) + (\gamma(U))^c(x) \not> 1$. It follows that p_x^{λ} is not a fuzzy γ-accumulation point of D and hence we have a contradiction.

We conclude this chapter with the following result on fuzzy γ-continuous and compactness.

Theorem 3.5.12: Let f be a mapping of a fts (X,T) into another fts (Y,T'), and γ an operation on T'. If (X,T) is fuzzy compact and f is fuzzy γ-continuous, then f(X) is fuzzy γ-compact.

Proof: Let $C = \{V_i : i \in I\}$ be an open cover of f(X). By the fuzzy γ-continuity, the set $D = \{U_i \in T : f(U_i) \subseteq \gamma(V_i)\}$ for some $V_i \in C$ } is open cover of X. Since X is fuzzy compact, then $X \subseteq \bigcup\{U_i : i = 1,2\cdots n\}$ for some $U_1, U_2 \cdots U_n \in D$. For each {1, 2, …n}, we can find $V_i \in C$ such that $f(U_i) \subseteq \gamma(V_i)$. Therefore

$f(X) \subseteq f(\bigcup\{U_i : i = 1,2,\cdots n\}) = \bigcup\{f(U_i) : i = 1,2,\cdots n\} \subseteq \bigcup\{\gamma(V_i) : i = 1,2,\cdots n\}$ and so f(X) is fuzzy γ-compact.