3 (Sem–3 / CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT-HC-3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1\times10=10$

   (a) Find $\lim\limits_{x\to2}\dfrac{x^3-4}{x^2+1}$

   (b) Is the function $f(x)=x\sin\left(\dfrac{1}{x}\right)$ continuous at $x=0$?

   (c) Write the cluster points of $A=(0,1)$.

(d) If a function $f: (a, \infty) \to \mathbb{R}$ is such that $\lim_{x \to \infty} xf(x) = L$, where $L \in \mathbb{R}$, then $\lim_{x \to \infty} f(x) = ?$

(e) Write the points of continuity of the function $f(x) = \cos\sqrt{1+x^2}$, $x \in \mathbb{R}$.

(f) "Every polynomial of odd degree with real coefficients has at least one real roof." Is this statement true **or** false ?

(g) The derivative of an even function is ———— function. *(Fill in the blank)*

(h) Between *any two* roots of the function $f(x) = \sin x$, there is at least ———— root of the function $f(x) = \cos x$. *(Fill in the blank)*

(i) If $f(x) = |x^3|$ for $x \in \mathbb{R}$, then find $f'(x)$ for $x \in \mathbb{R}$.

(j) Write the number of solutions of the equation $\ln(x) = x - 2$.

2. Answer

(a) Sho exis

(b) Let $f(x$ at x con

(c) Sho con

(d) Giv a fu but ever

(e) Sup $f(x$ $f(0$

ıe following questions : 2×5=10

v that $\lim_{x\to 0}(x+sgn(x))$ does not

be defined for all $x\in\mathbb{R}$, $x\neq 3$ by

$=\frac{x^2+x-12}{x-3}$. Can f be defined

$=3$, in such a way that f is

nuous at this point ?

that $f(x)=x^2$ is uniformly

nuous on $[0, a]$, where $a>0$.

an example with justification that

ction is 'continuous at every point

whose derivative does not exist

where'.

pose $f: \mathbb{R}\to\mathbb{R}$ be defined by

$=x^2 \sin\frac{1}{x^2}$, for $x\neq 0$ and

$=0$. Is f' bounded on $[-1,1]$?

3. Answer ***any four*** parts : 5×4=20

(a) If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then prove that f is bounded on some neighbourhood of c.

(b) Let $f(x) = |2x|^{-\frac{1}{2}}$ for $x \neq 0$. Show that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = +\infty.$$

(c) Show that the function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(d) Give an example to show that the product of two uniformly continuous function is not uniformly continuous on $\mathbb{R}$.

(e) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If f' is positive on $[a, b]$, then prove that f is strictly increasing on $[a, b]$.

(f) Evaluate —

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

4. Answer ***any four*** parts : 10×4=40

(a) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Prove that the following are equivalent.

(i) $\lim_{x \to c} f(x) = l$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $x \in \mathbb{N}$, the sequence $(f(x_n))$ converges to l. 10

(b) (i) Give examples of functions f and g such that f and g do not have limits at a point c but such that both $f+g$ and fg have limits at c. 6

(ii) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a cluster point of A. If $\lim_{x \to c} f(x)$ exists and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) = |fx|$, Proof that

$$\lim_{x \to c} |f|(x) = \left| \lim_{x \to c} f(x) \right| \qquad 4$$

(c) Prove that the rational functions and the sine functions are continuous on $\mathbb{R}$. 10

(d) (i) Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. Prove that the set $f(I)$ is an interval. 5

(ii) Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 5

(e) State and prove maximum-minimum theorem. 2+8=10

(f) (i) If $f: I \to \mathbb{R}$ has derivative at $c \in I$, then prove that f is continuous at c. Is the converse true ? Justify. 6

(ii) If r is a rational number, let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine those values of r for which $f'(0)$ exists. 4

(g) State and prove Mean value theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(h) State and prove Taylor's theorem.
