2019

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

Symbols have usual meaning

1. Answer the following questions : 1×7=7

(a) Write down a sufficient condition for the equality of f_{xy} and f_{yx}.

(b) Give an example of a discontinuous function which in Riemann integrable.

(c) If P^* is a refinement of a partition P of a bounded function f, then write down the relations between $U(P, f)$, $U(P^*, f)$, $L(P, f)$, $L(P^*, f)$.

(d) Define pole of order n of a complex valued function $f(z)$.

(e) A function $f(z) = u(x, y) + iv(x, y)$ is defined such that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. State whether f is analytic or not.

(f) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a region R. Prove that $\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2$.

(g) Find the fixed points of the transformation $w = z + 5$.

2. Answer the following questions : 2×4=8

(a) Show that

$$\lim_{x\to 0} \lim_{y\to 0} f(x, y) = \lim_{y\to 0} \lim_{x\to 0} f(x, y)$$

but $\lim_{(x, y)\to(0, 0)} f(x, y)$ does not exist, where

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$
$$= 0, \quad (x, y) = (0, 0)$$

(b) Prove that the improper integral

$$\int_a^b \frac{dx}{(x-a)^n}$$

converges if and only if $n<1$.

(c) Let C be the curve in the xy-plane defined by $3x^2y-2y^3=5x^4y^2-6x^2$. Find a unit vector normal to C at $(1, -1)$.

(d) Show that

$$\nabla \equiv \frac{\partial}{\partial x}+i\frac{\partial}{\partial y}=2\frac{\partial}{\partial \bar{z}}$$

3. Answer any *three* parts : 5×3=15

(a) Show that the function

$$f(x, y)=\frac{x^2y}{x^4+y^2}, \quad x^2+y^2\neq 0$$
$$=0, \quad x=y=0$$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

(b) Prove that a bounded function f is integrable on $[a, b]$ iff for every $\varepsilon>0$, there exists a partition P of $[a, b]$ such that $U(P, f)-L(P, f)<\varepsilon$.

(c) Prove that every absolutely convergent improper integral is convergent.

(d) Given, $u = e^{-x}(x\sin y - y\cos y)$, find v such that $f(z) = u + iv$ is analytic.

(e) Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by *(i)* $z = t^2 + it$ and *(ii)* the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

4. Answer any *one* part : 10

(a) *(i)* Show that $f(xy, z - 2x) = 0$, f is differentiable and $f_v \neq 0$, where $v = z - 2x$ satisfies the equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$$

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(ii) Show that the function

$$f(x, y) = y^2 + x^2y + x^4$$

has a minimum at (0, 0). 5

(b) (i) Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists if and only if m, n both are positive. 5

(ii) Show that the integral

$$\int_0^1 \frac{\sin(1/x)}{x^p}dx, \; p>0$$

is absolutely convergent for $p<1$. 5

5. Answer any *one* part : 10

(a) (i) The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w. Prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$ 5

(ii) Prove that if f and g are Riemann integrable on $[a, b]$, then $f+g$, $f-g$ are also Riemann integrable on $[a, b]$. 5

(b) (i) Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is Riemann integrable in $[0, 3]$. 5

(ii) Prove that if a function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then $\int_a^b f\,dx = F(b) - F(a)$. 5

6. Answer any *one* part : 10

(a) *(i)* Prove that if

$$w = f(z) = u(x, y) + iv(x, y)$$

is analytic, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ 5

(ii) Let $u(x, y) = \alpha$ and $v(x, y) = \beta$, where u and v are the real and imaginary parts of an analytic function $f(z)$ and α, β are the constants, represent two families of curves. Prove that if $f'(z) \neq 0$, then the families are orthogonal. 5

(b) *(i)* Let $f(z)$ be analytic inside and on a circle C of radius r and centre at $z = a$. Then prove that

$$f^{(n)}(a) \leq \frac{Mn!}{r^n}, \; n = 0, 1, 2, \ldots$$

where M is a constant such that $|f(z)| < M$ on C and $f^{(n)}(a)$ represents n-th derivative of $f(z)$ at $z = a$. 5

(ii) Let the rectangular region R in the z-plane be bounded by $x = 0$, $y = 0$, $x = 2$, $y = 1$. Determine the region R' of the w-plane into which R is mapped under the transformation

1. $w = z + (1 - 2i)$
2. $w = \sqrt{2}e^{i\pi/4}z$ 2+3=5

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